International Research Journal of Advanced Engineering and Science
ISSN (Online): 2455-9024

Fermat-Like Equations Near Zero

Mukesh Kumar[1], Rambha Kishor[2]
[1]Department of Mathematics, Magadh University, Bodh Gaya, Bihar, India-824234
[2]Department of Mathematics, Magadh University, Bodh Gaya, Bihar, India-824234

Abstract: *By Fermat's Last Theorem, the equation $x^n + y^n = z^n$ has no positive integer solution for all natural number $n > 2$. In contrast, the partition regularity of $x^2 + y^2 = z^2$ over the positive integers remains open. In this paper, we study the partition regularity of Fermat-type equation $x^n + y^n = z^n$ over the positive reals near zero, together with its infinitary extensions. Our results develop a Ramsey theoretic framework for Fermat-type equations near zero and reveal connections with the algebra of ultrafilters.*

Keywords- *Algebra of the Stone–Čech compactification, Ultrafilters near zero, IP-Sets, Idempotent Ultrafilters near zero.*

I. Introduction

Ramsey theory studies the persistence of combinatorial configurations under finite colorings. One of the fundamental notions in this area is partition regularity. An equation is called partition regular if every finite coloring of its underlying set contains a monochromatic solution.

The Stone–Čech compactification of a discrete semigroup provides a powerful algebraic framework for studying partition regularity. Let $(S,\cdot)$ be an infinite discrete semigroup. Now the points of βS are taken to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, let $\bar{A} = \{p \in \beta S: A \in p\}$. Then the set $\{\bar{A}: A \subseteq S\}$ will become a basis for a topology on βS. The operation $'\cdot'$ on S can be extended to the Stone–Čech compactification βS of S so that $(\beta S,\cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p: \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x: \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S: x^{-1}A \in q\} \in p$, Where $x^{-1}A = \{y \in S: x \cdot y \in A\}$. According to Ellis's theorem, if T is a non-empty compact right topological semigroup, then there exists an element $e \in T$ such that $e^2 = e$. The existence of idempotents within the compact space βS yields a wide array of powerful applications in Ramsey Theory and Combinatorics.

A nonempty subset I of a semigroup $(T,\cdot)$ is called a left ideal of T if $T \cdot I \subseteq I$, a right ideal if $I \cdot T \subseteq I$, and a two-sided ideal (or simply an ideal) if it is both a left and right ideal.

A minimal left ideal is a left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and smallest ideal. Any compact Hausdorff right topological semigroup $(T,\cdot)$ has a unique two-sided ideal
$K = \bigcup\{L: L \text{ is a minimal left ideal}\} =$
$\bigcup\{R: R \text{ is a minimal right ideal}\}$.
Given a minimal left ideal L and a minimal right ideal R of T, $L \cap R$ is a group, and in particular $K(T)$ contains an idempotent.

An idempotent that belongs to $K(T)$ is called a minimal idempotent. Given any subset $J \subseteq T$, we use the notation $E(J)$ to denote the set of all idempotents in J.

In [HLS], Hindman, Leader, and Strauss introduced ultrafilters near zero and studied various Ramsey-theoretic results in this setting. In this article, we study the partition regularity of a Fermat-like equation near zero. Let $\mathbb{R}^+ = (0,\infty)$ be the set of all positive real numbers. We know that $\mathbb{R}^+$ forms a semigroup under usual addition $+$. For each $k \in \mathbb{N}$, let us define a binary operation $\oplus_k$ as $x \oplus_k y = \sqrt[k]{x^k + y^k}$. Note that $\oplus_1 = +$.

Proposition 1.1 For each $k \in \mathbb{N}$, $(\mathbb{R}^+ \oplus_k)$, is a semigroup.

Proof. Let $k \in \mathbb{N}$ be a fixed positive integer. Then
$x \oplus_k y = \sqrt[k]{x^k + y^k} \in \mathbb{R}^+$ for every $x, y \in \mathbb{R}^+$.
We only need to show that is associative. To this end, let $x, y, z \in \mathbb{R}^+$. Then

$$\begin{aligned}(x \oplus_k y) \oplus_k z &= \left(\sqrt[k]{x^k + y^k}\right) \oplus_k z \\ &= \sqrt[k]{\left(\sqrt[k]{x^k + y^k}\right)^k + z^k} \\ &= \sqrt[k]{x^k + y^k + z^k} \\ &= \sqrt[k]{x^k + (y^k + z^k)} \\ &= x \oplus_k (y \oplus_k z).\end{aligned}$$

And so, $(x \oplus_k y) \oplus_k z = x \oplus_k (y \oplus_k z)$ for $x, y, z \in \mathbb{R}^+$.
Due to Proposition 1.1,
$(\beta\mathbb{R}^+, \oplus_k)$, is a compact right topological semigroup for each $k \in \mathbb{N}$. By Corollary 5.9 of [HLS], we obtain the following theorem. Throughout this section, we use the notation $P_f(X)$ to denote the set of all finite subsets of X.

Theorem 1.2 Let $n \in \mathbb{N}$ and let $S = \bigcup_{i=1}^{n} E_i$. Then, for each $k \in \mathbb{N}$, there exist $i \in \{1, 2, \ldots, n\}$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^+$ such that $\oplus_k$-FS $(\langle x_n \rangle_{n=1}^{\infty}) = \left\{\left(\sum_{n \in F} x_n^k\right)^{1/k}: F \in P_f(\mathbb{N})\right\} \subseteq E_i$.

As a Corollary of Theorem 1.2, we can say that for each $k \in \mathbb{N}$, $Z^k = X^k + Y^k$ is partition regular. In this article we will prove the near zero version of these results.

II. Algebra of Ultrafilters near Zero

In this section, we investigate the algebra of ultrafilters near zero with respect to $\oplus_k$ operations for each $k \in \mathbb{N}$. We begin by introducing the foundational definition.

Definition 2.1 Let $O(\mathbb{R}^+) = \{p \in \beta\mathbb{R}^+: (0,\varepsilon) \in p \text{ for every } \varepsilon > 0\}$. The algebraic substructure of $O(\mathbb{R}^+)$is given by the following theorem.

Theorem 2.2 For each $k \in \mathbb{N}$, $O(\mathbb{R}^+)$ is a compact right topological semigroup with respect to the binary operation $\oplus_k$ on $\mathbb{R}^+$extended to $\beta\mathbb{R}^+$.

Mukesh Kumar and Rambha Kishor, "Fermat-Like Equations Near Zero," International Research Journal of Advanced Engineering and Science, Volume 11, Issue 4, pp. 43-45, 2026.

International Research Journal of Advanced Engineering and Science
ISSN (Online): 2455-9024

Proof. Clearly, by the definition, $O(\mathbb{R}^+) = \bigcap_{\varepsilon>0} \mathrm{cl}_{\beta\mathbb{R}^+}(0,\varepsilon)$ is a closed subset of $\beta\mathbb{R}^+$, and so is a nonempty compact subset of $\beta\mathbb{R}^+$. Now it is enough to show that $O(\mathbb{R}^+)$ is a subsemigroup of $(\beta\mathbb{R}^+ \oplus_k)'$. Let $p, q \in O(\mathbb{R}^+)$ and $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{2^k}$. Then for each $x \in (0,\delta)$, $x \oplus_k y \in (0,\varepsilon)$ for all $y \in (0,\delta)$, and so $(0,\delta) \subseteq x^{-1} \oplus_k (0,\varepsilon) = \{y \in \mathbb{R}^+ : x \oplus_k y \in (0,\varepsilon)\}$.
Hence $(0,\delta) \subseteq \{x \in \mathbb{R}^+ : x^{-1} \oplus_k (0,\varepsilon) \in q\}$, and thus $\{x \in \mathbb{R}^+ : x^{-1} \oplus_k (0,\varepsilon) \in q\} \in p$.
Therefore, $(0,\varepsilon) \in p \oplus_k q$, which implies $p \oplus_k q \in O(\mathbb{R}^+)$. for all $p, q \in O(\mathbb{R}^+)$, as required.

In the following lemma, $t.p$ refers to multiplication in $(\beta\mathbb{R}^+, \cdot)$ for all $t \in \mathbb{R}^+$.

Lemma 2.3 Let $k \in \mathbb{N}$ and let p be an idempotent in $(\beta\mathbb{R}^+, \oplus_k)'$. For each $t \in \mathbb{R}^+$, $t \cdot p$ is also an idempotent in $\beta\mathbb{R}^+$. Furthermore, if p is an idempotent in $(O(\mathbb{R}^+) \oplus_k)'$ then $t \cdot p$ is an idempotent in $(O(\mathbb{R}^+) \oplus_k)'$ for all $t \in \mathbb{R}^+$.

Proof. For $t \in \mathbb{R}^+$, define the map $f_t : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f_t(x) = t \cdot x$. Let $\tilde{f}_t : \beta\mathbb{R}^+ \to \beta\mathbb{R}^+$ be the continuous extension of f_t. Since f_t is a semigroup isomorphism, so is $\tilde{f}_t$. Therefore, $t \cdot p \& = \tilde{f}_t(p) = \tilde{f}_t(p \oplus_k p) = \tilde{f}_t(p) \oplus_k \tilde{f}_t(p) = t \cdot p \oplus_k t \cdot p$ and thus $t \cdot p$ is an idempotent in $(\beta\mathbb{R}^+ \oplus_k)'$. If $p \in O(\mathbb{R}^+)$, for $\varepsilon > 0$, $(0, \varepsilon/t) \in p$. Then $(0,\varepsilon) \in t \cdot p$. and so $t \cdot p \in O(\mathbb{R}^+)$. Therefore, if p is an idempotent in $(O(\mathbb{R}^+) \oplus_k)'$, so is $t \cdot p$.

Following theorem studies the multiplicative structure of idempotent in $(\beta\mathbb{R}^+, \oplus_k)'$ for all $k \in \mathbb{N}$.

Theorem 2.4 For each $k \in \mathbb{N}$, $\Gamma_k = \mathrm{cl}_{\beta\mathbb{R}^+} E(\beta\mathbb{R}^+ \oplus_k)'$ and $\Gamma_k^0 = \mathrm{cl}_{\beta\mathbb{R}^+} E(O(\mathbb{R}^+) \oplus_k)'$ are left ideals of $(\beta, \mathbb{R}^+ \cdot)$.

Proof. Let $p \in \Gamma_k$. We claim that $\mathbb{R}^+ \cdot p \subseteq \Gamma_k$, so that $\mathrm{cl}_{\beta\mathbb{R}^+}(\mathbb{R}^+ \cdot p) = \beta\mathbb{R}^+ \cdot p \subseteq \Gamma_k$. Let $t \in \mathbb{R}^+$and let $A \in t \cdot p$. Then $t^{-1}A \in p$. Thus, choose $q \in E(\beta\mathbb{R}^+ \oplus_k)'$ such that $t^{-1}A \in q$. Then $A \in t \cdot q$ and, by Lemma 2.3, $t \cdot q \in E(\beta\mathbb{R}^+ \oplus_k)'$, as required.

The proof for Γ_k^0 is similar to that for Γ_k.

3. $\oplus_k$-IP Sets Near Zero

In this section, we prove Hindman's theorem for $\oplus_k$ operations for each $k \in \mathbb{N}$ and deduce the partition regularity of Fermat-like equations.

Definition 3.1

For each $k \in \mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$, define $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) = \left\{ \left(\sum_{n \in F} x_n^k \right)^{1/k} : F \in P_f(\mathbb{N}) \right\}$.

Similarly, for any finite sequence $\langle x_n \rangle_{n=1}^{m}$, with $m \in \mathbb{N}$, define $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{m}) = \left\{ \left(\sum_{n \in F} x_n^k \right)^{1/k} : F \in \{1, \dots, m\} \right\}$. We now define $\oplus_k$-IP sets and $\oplus_k$-IP sets near zero for each $k \in \mathbb{N}$.

Definition 3.2

For each $k \in \mathbb{N}$:

(a) A subset $A \subseteq \mathbb{R}^+$is said to be an $\oplus_k$-IP set if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$such that $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

(b) A subset $A \subseteq \mathbb{R}^+$is said to be an $\oplus_k$-IP set near zero if there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$such that $\sum_{n=1}^{\infty} x_n$ converges and $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

Theorem 3.3

Let $k \in \mathbb{N}$ and $p \in E(O(\mathbb{R}^+) \oplus_k)'$. Let $A \in p$. Then there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$such that $\sum_{n=1}^{\infty} x_n$ is convergent and $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

Proof.

Let $A^* = \{x \in A : x^{-1}A \in p\}$. Then $A^* \in p$. Choose $x_1 \in A^* \cap \left(0, \frac{1}{2}\right)$. Let $n \in \mathbb{N}$ and assume we have chosen $\langle x_t \rangle_{t=1}^{n}$ such that $x_t < \frac{1}{2^t}$ for $1 \le t \le n$ and $E = \oplus_k\text{-FS}(\langle x_t \rangle_{t=1}^{n}) \subseteq A^*$. Then E is finite and, for each $a \in E$, we have, by [H.S. Lemma 4.14], $a^{-1}A^* \in p$. Hence $\bigcap_{a \in E} a^{-1}A^* \in p$. Pick $x_{n+1} \in A^* \cap \bigcap_{a \in E} a^{-1}A^* \cap \left(0, \frac{1}{2^{n+1}}\right)$. Then $x_{n+1} \in A^*$ and, given $a \in E$, $a \oplus_k x_{n+1} \in A^*$. Hence $\oplus_k\text{-FS}(\langle x_t \rangle_{t=1}^{n+1}) \subseteq A^*$, and also $x_t < \frac{1}{2^t}$ for $1 \le t \le n$.

Proof. Since $O(\mathbb{R}^+ \oplus_k)'$ is a compact right topological semigroup, choose $p \in E(O(\mathbb{R}^+) \oplus_k)'$. Choose $i \in \{1, 2, \dots, r\}$ such that $A_i \in p$. For each $\varepsilon > 0$, $A_i \cap (0,\varepsilon) \in \phi$, and so $A_i \cap (0,\varepsilon)$ is an $\oplus_k$-IP-set near zero. Hence choose a sequence $\langle x_n \rangle_{n \ge 1}$ in $\mathbb{R}^+$such that $\sum_{n=1}^{\infty} x_n$ is convergent and $\oplus_k\text{-FS}(\langle x_n \rangle_{n \ge 1}) \subseteq A_i \cap (0,\varepsilon)$ as required.

Corollary 3.6: For each $k \in \mathbb{N}$, $x^k + y^k = z^k$ is partition regular near zero.

Proof. Let $r \in \mathbb{N}$ and $\mathbb{R}^+ = \bigcup_{i=1}^{r} A_i$ be an r-coloring of $\mathbb{R}^+$. Let $\varepsilon > 0$ be given. By Corollary 3.5, choose $i \in \{1, 2, \dots, r\}$ and $x, y \in \mathbb{R}^+$such that $\{x, y, x \oplus_k y\} \subseteq A_i \cap (0,\varepsilon)$. Let $z = x \oplus_k y$. Then $z = (x^k + y^k)^{1/k}$ and so $z^k = x^k + y^k$. Hence $\{x, y, z\} \subseteq A_i \cap (0,\varepsilon)$.

Theorem 3.5. Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{R}^+$. Then

(a) A is an $\oplus_k$-IP set if and only if there exists $p \in E(\beta\mathbb{R}^+ \oplus_k)'$ such that $A \in p$.

(b) A is an $\oplus_k$-IP set near zero if and only if there exists $p \in E(O(\mathbb{R}^+) \oplus_k)'$ such that $A \in p$.

Proof.

(a) Let $S = (\mathbb{R}^+ \oplus_k)'$. If A is an $\oplus_k$-IP set, then by [H.S., Lemma 5.12], there exists an idempotent $p \in E(\beta\mathbb{R}^+ \oplus_k)'$ such that $A \in p$.

Conversely, if $A \in p$, where $p \in E(\beta\mathbb{R}^+ \oplus_k)'$, then by Theorem 5.8, there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$such that $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$, and hence A is an $\oplus_k$-IP set.

(b) Let Abe an $\oplus_k$-IP set near zero. Then there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$such that $\sum_{n=1}^{\infty} x_n$ converges and $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

Let $T = \bigcap_{m=1}^{\infty} \mathrm{cl}_{\beta\mathbb{R}^+}\left(\oplus_k\text{-FS}(\langle x_n \rangle_{n=m}^{\infty})\right)$. Then, by [H.S., Lemma 5.11], T is a subsemigroup of $(\beta\mathbb{R}^+ \oplus_k)'$. Since $\sum_{n=1}^{\infty} x_n$ is convergent, $\oplus_k\text{-FS}(\langle x_n \rangle_{n=m}^{\infty}) \cap (0,\varepsilon) \neq \emptyset$ for each $m \in \mathbb{N}$ and $\varepsilon > 0$. Thus $T \cap O^*(\mathbb{R}^+) \neq \emptyset$. Hence $T \cap O(\mathbb{R}^+)$ is a subsemigroup of $(O(\mathbb{R}^+) \oplus_k)'$. Choose an idempotent $p \in E(O(\mathbb{R}^+) \oplus_k)'$ such that $p \in T$. Therefore, $A \in p$ for some $p \in E(O(\mathbb{R}^+) \oplus_k)'$, as required.

Corollary 3.5. Let $k, r \in \mathbb{N}$ and let $\mathbb{R}^+ = \bigcup_{i=1}^{r} A_i$. Then there exists $i \in \{1, 2, \dots, r\}$ such that, for any $\varepsilon > 0$, there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ With $\sum_{n=1}^{\infty} x_n$, convergent and $\oplus_k\text{-FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$.

Proof. Since $O(\mathbb{R}^+ \oplus_k)'$ is a compact right topological semigroup, choose $p \in E(O(\mathbb{R}^+) \oplus_k)'$. Choose $i \in \{1, 2, \dots, r\}$

Mukesh Kumar and Rambha Kishor, "Fermat-Like Equations Near Zero," International Research Journal of Advanced Engineering and Science, Volume 11, Issue 4, pp. 43-45, 2026.

such that $A_i \in p$. For each $\varepsilon > 0$, $A_i \cap (0, \varepsilon) \in \phi$, and so $A_i \cap (0, \varepsilon)$ is an $\bigoplus_k$-IP-set near zero. Hence choose a sequence $\langle x_n \rangle_{n\geq 1}$ in $\mathbb{R}^+$such that $\sum_{n=1}^{\infty} x_n$ is convergent and $\bigoplus_k$-FS $(\langle x_n \rangle_{n\geq 1}) \subseteq A_i \cap (0, \varepsilon)$ as required.

Corollary 3.6: For each $k \in \mathbb{N}$, $x^k + y^k = z^k$ is partition regular near zero.

Proof. Let $r \in \mathbb{N}$ and $\mathbb{R}^+ = \bigcup_{i=1}^{r} A_i$ be an r-coloring of $\mathbb{R}^+$. Let $\varepsilon > 0$ be given. By Corollary 3.5, choose $i \in \{1, 2, \dots, r\}$ and $x, y \in \mathbb{R}^+$such that $\{x, y, x \bigoplus_k y\} \subseteq A_i \cap (0, \varepsilon)$. Let $z = x \bigoplus_k y$. Then $z = (x^k + y^k)^{1/k}$ and so $z^k = x^k + y^k$. Hence $\{x, y, z\} \subseteq A_i \cap (0, \varepsilon)$.

III. Conclusion

In this paper, we have studied the algebra of ultrafilters for the semigroup $(\mathbb{R}^+ \bigoplus)$, where the operation is defined as $x \bigoplus_k y = \sqrt[k]{x^k + y^k}$ for each fixed $k \in \mathbb{N}$.

In particular, we have proved the Hindman theorem near zero for this semigroup. One can study the Central Sets Theorem and establish new partition regularity of different Diophantine equations over $\mathbb{R}^+$ for further study.

References

[1] N. Hindman, and D. Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications*, Second Edition, de Gruyter, Berlin, 2012.

[2] N. Hindman, I. Leader, *The Semigroup of Ultrafilters Near Zero*, Semigroup Forum **59** (1999), 33–55.