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D-Sets in Commutative Adequate Partial Semigroup

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Abstract— *Biswas, Bose, and Patra introduced the notion of D-sets in arbitrary semigroups in 2020. Inspired by their work, we extend the concept to arbitrary adequate partial semigroups and explore its algebraic, dynamical, and combinatorial structure. We establish dynamical and combinatorial characterizations of D-sets, thereby extending the theory to the partial-semigroup setting and uncovering further connections with ultrafilter algebra, topological dynamics, and Ramsey theory.*

Keywords- *D-sets, Adequate Particle Semigroup, Strong Følner Condition (SFC), Ultrafilters*

I. INTRODUCTION

Biswas, Bose and Patra [1] introduced the notion of D-sets in arbitrary semigroups, based on the concept of density in arbitrary semigroups developed by Hindman and Strauss [2]. We shall use the following notion of upper Banach density, as defined in [2, Definition 2.1(c)], to introduce D-sets in arbitrary semigroups.

For a set X, we denote by $P_f(X)$ the collection of all finite nonempty subsets of X.

Definition 2.1

Let S be a semigroup, let $\mathcal{F} = \langle F_i\rangle_{i\in I}$ be a net in $P_f(S)$, and let $A \subseteq S$. Then the upper Banach density of A with respect to $\mathcal{F}$ is $d^*_{\mathcal{F}} = \sup\{\alpha : (\forall i_0 \in I)(\exists i > i_0)(\exists s \in S \cup \{1\})(| A \cap (F_i \cdot s) | \geq \alpha \cdot | F_i |)\}$.

We do not assume that S possesses an identity element. Accordingly, for each $i \in I$, the notation $F_i \cdot 1$ is to be understood simply as F_i.

The concept of D-sets in arbitrary semigroup S admits a natural algebraic formulation within βS, the Stone–Čech compactification of the discrete semigroup S.

Let $(S,\cdot)$be any discrete semigroup and βS, the set of ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Now for any $A \subseteq S$, let $\bar{A} = \{p \in \beta S : A \in p\}$. Then the collection $\{\bar{A} : A \subseteq S\}$ becomes a basis for a topology on βS. The operation "$\cdot$" on S can be extended to the Stone–Čech compactification βS of S so that $(\beta, S\,\cdot)$ becomes a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : xy \in A\}$.

A nonempty subset I of a semigroup $(T,\cdot)$ is called a left ideal of T if $T \cdot I \subseteq I$, a right ideal of T if $I \cdot T \subseteq I$, and a two-sided ideal (or simply, an ideal) if it is both a left and a right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and smallest ideal. Any compact Hausdorff right topological semigroup $(T,\cdot)$ has a smallest two-sided ideal:

$$K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\}$$

and

$$K(T) = \bigcup\{R : R \text{ is a minimal right ideal of } T\}$$

Ellis's theorem guarantees that every compact right topological semigroup must contain an idempotent. In particular, $K(T)$ contains an idempotent, which is called a minimal idempotent. For an elementary introduction to algebra of βS, see [9, Part-I].

We are now in a position to define D-sets in arbitrary semigroups. Following definition is defined in [1, Definition 2.2].

Definition 1.2 Let S be a semigroup, let $\mathcal{F}$ be a net in $P_f(S)$, and let $D^*_{\mathcal{F}} = \{p \in \beta S : (\forall A \in p)\ (d^*_{\mathcal{F}}(A) > 0)\}$. Then a subset A of S is a D-set with respect to $\mathcal{F}$ if it is contained in an idempotent of $D^*_{\mathcal{F}}$.

In [3], a notion of density was introduced for commutative adequate partial semigroups.

A partial semigroup [3, Definition 2.1] is a pair $(S,\cdot)$, where "$\cdot$" is a partially defined binary operation on S, such that for all $a, b, c \in S$, the expressions $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$ are either both defined and equal or both undefined.

Example 1.3 **(1)** Let $R = \{A, : A \text{ is an } m \times n \text{ integer matrix for some } m, n \in \mathbb{N}\}$ with the usual matrix multiplication. Considering M as an $m \times n$ matrix and N as an $m' \times n'$ matrix, we define

$$M \cdot N = \begin{cases} MN, & n = m', \\ \text{undefined}, & n \neq m'. \end{cases}$$

Then $(R,\cdot)$ is a partial semigroup.

(2) Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in a semigroup $(S,\cdot)$, and define $FP(\langle x_n\rangle_{n=1}^{\infty}) = \{\prod_{n\in F} x_n : F \in P_f(\mathbb{N})\}$, with the product taken in increasing order. Define

$$\left(\prod_{n\in F} x_n\right) \cdot \left(\prod_{n\in G} x_n\right) = \begin{cases} \prod_{n\in F\cup G} x_n\,, & \max F < \min G\,, \\ \text{undefined}, & \max F \geq \min G\,. \end{cases}$$

Then $(F, P(\langle x_n\rangle_{n=1}^{\infty})\ \cdot)$ is a partial semigroup.

Definition 1.4 ([4, Definition 2.1]) Let $(S,\cdot)$ be a partial semigroup.

(a) For $a \in S$, $\phi(a) = \{b \in S : a * b \text{ is defined}\}$.

(b) For $F \in P_f(S)$, $\sigma(F) = \bigcap_{a\in F} \phi(a)$.

(c) The partial semigroup $(S,\cdot)$ is adequate if and only if for every $F \in P_f(S)$, $\sigma(F) \neq \emptyset$.

(d) If S is adequate, then $\delta S = \bigcap_{F\in P_f(S)} \overline{\sigma(F)}$.

Clearly, $\delta S \subseteq \beta S$. For $x \in S$ and $A \subseteq S$, define $x^{-1}A = \{y \in \phi(x) : x \cdot y \in A\}$. The following lemma describes the algebraic structure of adequate partial semigroups.

Lemma 1.5 ([4, Lemma 2.7]) Let S be an adequate partial semigroup and $A \subseteq S$, then:
(a) If $x \in S$, $q \in \overline{\phi(x)}$, and $A \subseteq S$, then $A \in x \cdot q$ if and only if $x^{-1}A \in q$.
(b) If $p \in \beta S$, $q \in \delta S$, and $A \subseteq S$, then $A \in p \cdot q$ if and only if $\{x \in S: x^{-1}A \in q\} \in p$.
Theorem 1.6 ([4, Theorem 2.10]) If S is an adequate partial semigroup, then δS, equipped with the extended operation, is a compact right topological semigroup.
In view of [3, Definitions 3.3 and 3.7], the Følner and strong Følner conditions naturally extend to commutative adequate partial semigroups as follows.
Definition 1.7 Let S be a commutative adequate partial semigroup.
(i) S satisfies the Følner Condition (FC) if and only if

$$\left(\forall H \in P_f(S)\right)(\forall \varepsilon > 0)\left(\exists K \in P_f(\sigma(H))\right)(\forall s \in H)(|sK \backslash K| < \varepsilon|K|).$$

(ii) S satisfies the Strong Følner Condition (SFC) if and only if

$$\left(\forall H \in P_f(S)\right)(\forall \varepsilon > 0)\left(\exists K \in P_f(\sigma(H))\right)(\forall s \in H)(|K \backslash sK| < \varepsilon|K|).$$

It is immediate that, in a commutative adequate partial semigroup, SFC implies FC. We now present the corresponding notion of density for commutative partial semigroups satisfying the Strong Følner Condition, as given in [3, Definition 3.5].
Definition 1.8 Let S be a commutative adequate partial semigroup which satisfies (SFC).

(a) For $A \subseteq S$, the Følner density of A is defined by

$$d_r(A) = \sup\left\{\alpha \in [0,1]: \left(\forall H \in P_f(S)\right)(\forall \varepsilon > 0)\left(\exists K \in P_f(\sigma(H))\right)(\forall s \in H)(|K \backslash sK| < \varepsilon|K|) \text{ and } |A \cap |K|| \geq \alpha|K|\right\}.$$

(b) $\Delta_r(S) = \{p \in \delta S: for\ all\ A \in p, d_r(A) > 0\}$.

We now state the following two theorems from **[3, Lemma 4.1]** and **[3, Lemma 4.2]**, respectively.
Theorem 1.9 Let S be a commutative adequate partial semigroup satisfying (SFC). Then $\bar{A} \cap \Delta_r(S) \neq \emptyset$ if and only if $d_r(A) > 0$.
Theorem 1.10 Let S be a commutative adequate partial semigroup satisfying (SFC). Then $\Delta_r(S)$ is a left ideal of δS. Suppose there exists $b \in \mathbb{N}$ and for all $H \in P_f(S)$, there exists $K \in P_f(\sigma(H))$ such that for all $x \in \sigma(K)$, ρ_x is at most b-to-1 and $Kx \in P_f(\sigma(H))$, then $\Delta_r(S)$ is a right ideal of δS.
Now we are in a position to define the notion of D-sets for commutative adequate partial semigroups.
Definition 1.11 Let S be a commutative adequate partial semigroup satisfying the Strong Følner Condition (SFC). Then a subset A of S is a D-set if it is contained in an idempotent of $\Delta_r(S)$.
Authors of **[5]** established a dynamical characterization of central sets in adequate partial semigroups. In Section 2, motivated by their work, we establish a dynamical characterization of D-sets in commutative adequate partial semigroups. Also, in [6], the authors established a combinatorial characterization of C-sets near zero, a distinguished class of combinatorially large sets. Inspired by this development, Section 3 is devoted to establishing a corresponding combinatorial characterization of D-sets in commutative adequate partial semigroups.

II. Dynamical Characterization

We begin this section by recalling the definition of a dynamical system over an adequate partial semigroup.
Definition 2.1 ([5, Definition 2.1]) Let S be an adequate partial semigroup. A dynamical system is a pair $(X, \langle T_s \rangle_{s \in S})$ such that

1. X is a compact Hausdorff space;
2. For each $s \in S$, the map $T_s: X \to X$ is continuous;
3. For all $s, t \in S$, one has $T_s T_t = T_{st}$ whenever $t \in \phi(s)$.

The following definition [7, Definition 2.1] is useful for the dynamical characterization of D-sets in commutative adequate partial semigroup.
Definition 2.2 Let S be a nonempty discrete space and let $\mathcal{K}$ be a filter on S.

1. $\bar{\mathcal{K}} = \{p \in \beta S: \mathcal{K} \subseteq p\}$.
2. $\mathcal{L}(\mathcal{K}) = \{A \subseteq S: S \setminus A \notin \mathcal{K}\}$.

Theorem 3.20 of [8] establishes a bijective correspondence between filters on S and compact subspaces of βS via $\mathcal{K} \mapsto \bar{\mathcal{K}}$. The following theorem gives a sharper form of this correspondence.
Theorem 2.3 ([7, Theorem 2.2]) Let S be a nonempty discrete space and $\mathcal{K}$ be a filter on S. Then

1. $\bar{\mathcal{K}} = \{p \in \beta S: A \in \mathcal{L}(\mathcal{K})$ for all $A \in p\}$.
2. Let $\beta \subseteq \mathcal{L}(\mathcal{K})$be closed under finite intersections. Then there exists $p \in \beta S$ with $\beta \subseteq p \subseteq \mathcal{L}(\mathcal{K})$.

We shall use the following definition from **[9, Definition 4.1]**.
Definition 2.4 Let S be an adequate partial semigroup, and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system, x and y be two points in X, and $\mathcal{K}$ be a filter on S. The pair (x, y) is called jointly $\mathcal{K}$-recurrent if and only if for every neighbourhood U of y, we have

$$\{s \in S: T_s(x) \in U \text{ and } T_s(y) \in U\} \in \mathcal{L}(\mathcal{K}) = \{A \subseteq S: S \setminus A \notin \mathcal{K}\}.$$

The following theorem from **[9, Theorem 4.4]** provides a dynamical characterization of the sets belonging to idempotent ultrafilters on adequate partial semigroups.
Theorem 2.5 Let S be an adequate partial semigroup, let $\mathcal{K}$ be a filter on S such that $\bar{\mathcal{K}}$ is a compact subsemigroup of δS, and let $A \subseteq S$. Then A is a member of an idempotent in $\bar{\mathcal{K}}$ if and only if there exists a dynamical system $(X, \langle T_s \rangle_{s \in S})$, points $x, y \in X$, and a neighbourhood U of y such that

1. the pair (x, y) is jointly $\mathcal{K}$-recurrent; and
2. $A = \{s \in S: T_s(x) \in U\}$.

The following two lemmas are useful to establish the main result in this section.
Lemma 2.6 Let S be a commutative adequate partial semigroup satisfying Strong Følner Condition (SFC). Let $\mathcal{K} = \{A \subseteq S: d_r(S \setminus A) = 0\}$. Then $\mathcal{K}$ is a filter on S and $\bar{\mathcal{K}} = \Delta_r(S)$.

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Equivalently, $\Delta_r(S) = \bigcap_{A\in\mathcal{K}} \bar{A}$. Moreover, if A is a subset of S such that $d_r(S \setminus A) = 0$, then A is a D-set.

Proof. By [8, Definition 3.1], a nonempty collection $\mathcal{K}$ of subsets of S is a filter on S if

(a) $A \cap B \in \mathcal{K}$ for all $A, B \in \mathcal{K}$;

(b) For $A \subseteq B \subseteq S$ with $A \in \mathcal{K}$, $B \in \mathcal{K}$;

(c) $\phi \notin \mathcal{K}$.

Now, to prove (a), let $A, B \in \mathcal{K}$. Thus $d_r(S \setminus A) = d_r(S \setminus B) = 0$. By [3, Proposition 3.6], $d_r\big(S \setminus (A \cap B)\big) = d_r\big((S \setminus A) \cup (S \setminus B)\big) \leq d_r(S \setminus A) + d_r(S \setminus B)$, and therefore $d_r\big(S \setminus (A \cap B)\big) = 0$. i.e., $A \cap B \in \mathcal{K}$. For (b), let $A \in \mathcal{K}$ and $A \subseteq B \subseteq S$. Then $S \setminus B \subseteq S \setminus A$, and we have $d_r(S \setminus B) \leq d_r(S \setminus A)$. Thus $d_r(S \setminus B) = 0$, so $B \in \mathcal{K}$. Now, (c) is trivial to check because $d_r(S) = 1$. Hence, $\mathcal{K}$ is a filter on S.

We now show that $\overline{\mathcal{K}} = \Delta_r(S)$. To this end, let $p \in \beta S \setminus \Delta_r(S)$. So there exists $A \in p$ such that $d_r(A) = 0$ i.e., $S \setminus A \in \mathcal{K}$. As p is an ultrafilter, by [8, Theorem 3.6], $A \in p$ implies $S \setminus A \notin p$. Therefore, $S \setminus A \in \mathcal{K} \setminus p$, i.e., $\mathcal{K} \nsubseteq p, i.e., p \in \beta S \setminus \overline{\mathcal{K}}$. Therefore, we have $\beta S \setminus \Delta_r(S) \subseteq \beta S \setminus \overline{\mathcal{K}}$. The above inclusion together we have $\beta S \setminus \overline{\mathcal{K}} = \beta S \setminus \Delta_r(S)$, i.e., $\overline{\mathcal{K}} = \Delta_r(S)$. Also, $\overline{\mathcal{K}} = \bigcap_{A\in\overline{\mathcal{K}}} \bar{A}$, and equivalently we have $\Delta_r(S) = \bigcap_{A\in\mathcal{K}} \bar{A}$.

Also, by Theorem 1.10, $\Delta_r(S)$ is a compact left ideal of δS. Now, as $\Delta_r(S) = \bigcap_{A\in\mathcal{K}} \bar{A}$, for any idempotent $p \in \Delta_r(S)$, we have $p \in \bigcap_{A\in\mathcal{K}} \bar{A}$, i.e., $A \in p$ for every subset A of S such that $d_r(S \setminus A) = 0$. Hence the last conclusion follows.

Lemma 2.7 Let S be a commutative adequate partial semigroup satisfying (SFC). Let $\mathcal{K} = \{A \subseteq S: d_r(S \setminus A) = 0\}$. Let $(X, \langle T_s\rangle_{s\in S})$ be a dynamical system. Then a pair (x, y) in $X \times X$ is jointly $\mathcal{K}$-recurrent if and only if, for any neighbourhood U of y, $d_r(\{s \in S: T_s(x) \in U \text{ and } T_s(y) \in U\}) > 0$.

Proof. (x, y) is jointly $\mathcal{K}$-recurrent if and only if, for any neighbourhood U of y, $S \setminus \{s \in S: T_s(x) \in U \text{ and } T_s(y) \in U\} \notin \mathcal{K}$. But $S \setminus \{s \in S: T_s(x) \in U \text{ and } T_s(y) \in U\} \notin \mathcal{K}$ if and only if $d_r(\{s \in S: T_s(x) \in U \text{ and } T_s(y) \in U\}) > 0$.

We are now in a position to present the main result of this section, which is the dynamical characterization of D-sets in commutative adequate partial semigroups.

Theorem 2.8 Let S be a commutative adequate partial semigroup satisfying (SFC). Then A is a D-set if and only if there exists a dynamical system $(X, \langle T_s\rangle_{s\in S})$ with points x and y in X such that, for any neighbourhood U of y, $d_r(\{s \in S: T_s(x) \in U \text{ and } T_s(y) \in U\}) > 0$

and, for some neighbourhood U_y of y, $A = \{s \in S: T_s(x) \in U_y\}$.

Proof. Note that $\Delta_r(S) = \overline{\mathcal{K}}$, where $\mathcal{K} = \{A \subseteq S: d_r(S \setminus A) = 0\}$. By Lemma 2.6, Theorem 2.5 provides the desired dynamical characterization of D-sets whenever we replace the condition of joint $\mathcal{K}$-recurrence by its equivalent (in terms of density), which we proved in Lemma 2.6.

III. Combinatorial Characterization

In [10, Theorem 3.8], Hindman, Maleki and Strauss provide a combinatorial characterization of central sets. Also, in [6] the authors provide a combinatorial characterization of the near zero version of C-sets. In this section, we present a combinatorial characterization of D-sets in commutative adequate partial semigroups. For this purpose, we first recall the notion of trees, as introduced in [10, Definition 3.4 and 3.5], through the following two definitions. We shall use the standard notation $\omega = \{0, 1, 2, \ldots\}$ for the first infinite ordinal. Recall that every ordinal is identified with the set of its predecessors; in particular, $0 = \phi$ and $2 = \{0, 1\}$. We also use the notation $f\restriction_n$ for the restriction of a function f to n. For instance, if $f = \{(0,0), (1,4), (2,1), (3,10)\}$, then $f\restriction_3 = \{(0,0), (1,4), (2,1)\}$.

Definition 3.1 T is a tree in A if T is a set of functions and, for each $f \in T$, $\text{dom}(f) \in \omega$ and $\text{range}(f) \subseteq A$, and if $\text{dom}(f) = n > 0$, then $f\restriction_{n-1} \in T$. T is a tree if for some A, T is a tree in A.

Definition 3.2 **(1)** Let f be a function with $\text{dom}(f) = n \in \omega$ and let s be given. Then $f^\frown S = f \cup \{n, s\}$.

(2) Given a tree T and $f \in T$, $B_f = B_f(T) = \{s: f^\frown s \in T\}$.

(3) Let S be an adequate partial semigroup and let $A \subseteq S$. Then T is a $*$ tree in A if T is a tree in A and, for all $f \in T$ and for all $s \in B_f$, $B_{f\cap s} \subseteq s^{-1}B_f$

Lemma 3.3 ([11, lemma 4.3]) Let S be an adequate partial semigroup and let $p \in \delta S$. Then p is an idempotent if and only if, for each $A \in p$, there is a nonempty set T of functions such that:

(a) For all $f \in T$, $\text{dom}(f) \in \omega$ and $\text{range}(f) \subseteq A$.

(b) For all $f \in T$, $B_f(T) \in p$.

(c) For all $f \in T$ and all $x \in B_f(T)$, $B_{f^\frown x}(T) \subseteq x^{-1}B_f(T)$.

Lemma 3.4 Let S be an adequate partial semigroup and $\{\sigma(H): H \in \mathcal{P}_f(S)\} \subseteq \mathcal{A} \subseteq \mathcal{P}(S)$ has the finite intersection property. If, for each $A \in \mathcal{A}$ and each $x \in A$, there exists $B \in \mathcal{A}$ such that $B \subseteq x^{-1}A$, then $\bigcap_{A\in\mathcal{A}} \bar{A}$ is a subsemigroup of δS.

Proof Let $T = \bigcap_{A\in\mathcal{A}} \bar{A}$. Since $\mathcal{A}$ has the finite intersection property, $T \neq \emptyset$. Also, $T \subseteq \delta S$ because $\{\sigma(H): H \in \mathcal{P}_f(S)\} \subseteq \mathcal{A}$. Let $p, q \in T$ and let $A \in \mathcal{A}$. Give $x \in A$, there exists so $B \in \mathcal{A}$ such that $B \subseteq x^{-1}A$, and hence $x^{-1}A \in q$. Thus $A \subseteq \{x \in S: x^{-1}A \in q\}$ so $\{x \in S: x^{-1}A \in q\} \in p$ By Lemma 1.5 (b), $A \in p \cdot q$.

Theorem 3.5 Let S be a commutative adequate partial semigroup satisfying (SFC), and let $A \subseteq S$. Then the statements (a), (b), (c) are equivalent and implied by statement (d). If S is countable all statements are equivalent.

(a) A is a D-set.

(b) There is a $*$ tree T in A such that, for each $F \in \mathcal{P}_f(T)$, $d_r\left(\bigcap_{f\in F} B_f\right) > 0$.

(c) There is a downward directed family $\langle C_F\rangle_{F\in J}$ of subsets of A such that

(i) for each $F \in J$ and each $s \in C_F$, there exists $G \in J$ with $C_G \subseteq s^{-1}C_F$.

(ii) for each $F \in J$, $d_r(C_F) > 0$.

(d) There is a decreasing sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of subsets of A such that

(i) for each $n \in \mathbb{N}$ and each $s \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq s^{-1}C_n$.

(ii) for each $n \in \mathbb{N}$, $d_r(C_n) > 0$.

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Proof: $(a) \Rightarrow (b)$ Choose an idempotent $p \in \Delta_r(S)$ for which $A \in \phi$. By Lemma 3.3, pick a nonempty set $T$of functions such that
(a) for all $f \in T$, $\text{dom}(f) \in \omega$ and $\text{range}(f) \subseteq A$
(b) for all $f \in T$, $B_f(T) \in p$
(c) for all $f \in T$and all $x \in B_f(T)$, $B_{f \wedge x}(T) \subseteq x^{-1}B_f(T)$.
So, by the definition 3.2(3), $T$is a $*$-tree. Also, for each $F \in \mathcal{P}_f(T)$, $B_f(T) \in p$ for all $f \in F$ and so $\bigcap_{f\in F} B_f(T) \in p \in \Delta_r(S)$.
Thus, by Theorem we have that $d_r\left(\bigcap_{f\in F} B_f(T)\right) > 0$; as required.
$(b) \Rightarrow (c)$: Let Tbe given as guaranteed by (c). Let $J = \mathcal{P}_f(T)$ and for each $F \in J$, $C_F = \bigcap_{f\in F} B_f(T)$ Then $\langle C_F \rangle_{F\in J}$ is a downward directed family and $d_r(C_F) > 0$
for each $F \in J$. Let $F \in J$and let $s \in C_F$. Let $G = \{f \frown s: f \in F\}$. Now for each $f \in F$ we have$B_{f \wedge s} \subseteq s^{-1}B_f$. So, $C_G \subseteq s^{-1}C_F$.
$(c) \Rightarrow (a)$ Note that $\mathcal{A} = \{C_F : F \in J\} \cup \{\sigma(H) : H \in P_f(S)\}$ satisfies the finite intersection property. Also for each $A \in \mathcal{A}$ and each $x \in A$, there exists $B \in \mathcal{A}$ such that $B \subseteq x^{-1}A$. So, by Lemma 3.4, $M = \bigcap_{F\in J} \overline{C_F}$ is a compact subsemigroup of δS. Now, it suffices to show that $M \cap \Delta_r(S) \neq \phi$, as this ensures the existence of an idempotent in $M \cap \Delta_r(S)$.
Then as $M \subseteq \overline{A}$, and it follows from the definition that A is a D-set.
Now, given $F \in J$, since $d_r(C_F) > 0$, by Theorem 1.9, $\overline{C_F} \cap \Delta_r(S) \neq \phi$. Since $\langle C_F \rangle_{F\in J}$ is a downward directed family, so $M \cap \Delta_r(S) \neq \phi$.
$(\boldsymbol{d}) \Rightarrow (\boldsymbol{c})$**:** Trivial
Assume that S is countable. We shall show that (b) implies (d). Let T be the guaranteed by (b). So T is countable. Enumerate T as $\{f_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $C_n = \bigcap_{k=1}^{n} B_{f_k}(T)$. Then, for each $n \in \mathbb{N}$, $d_r(C_n) > 0$. Let $n \in \mathbb{N}$ and let $s \in C_n$. Pick some $m \in \mathbb{N}$ such that $(\{f_k \frown s : k \in \{1,2,3,\dots.m\}\}) \subseteq \{f_1, f_2, \dots, f_m\}$. Then $C_m \subseteq s^{-1}C_n$.

IV. Conclusion

In this article, we have defined D-sets using the algebra of ultrafilters and also established a dynamical and combinatorial characterization of D-sets. It is obvious from the definition that every central set is a D-set for any adequate partial semigroup satisfying SFC. Now, at the end of this paper, we may raise the following question:
4.1. Does every D-set satisfy the conclusion of the Central Sets Theorem?

References

1. S. Biswas, B. Bose, and S. K. Patra, *D-sets in arbitrary semigroups*, Topology and its Applications, 282 (2020), 107322.
2. N. Hindman, D. Strauss, *Density in arbitrary semigroups*, Semigroup Forum, 73(2) (2006), 273–300.
3. A. Ghosh, *Density in partial semigroups*.
4. N. Hindman, R. McCutcheon, *VIP Systems in partial semigroups*, Discrete Mathematics, 244 (2001), 45–70.
5. P. Debnath, S. Goswami, S. K. Patra, *Dynamical Characterization of Central Sets in Adequate Partial Semigroups*, Indagationes Mathematicae.
6. T. Bhattacharya, S. Chakraborty, S. K. Patra, *Combined algebraic properties of C^*-sets near zero*, Semigroup Forum, 100(3), 717–731.
7. J. H. Johnson, *A dynamical characterization of C-sets*, arXiv:1112.0715.
8. N. Hindman, D. Strauss, *Algebra in the Stone-Čech Compactification*, De Gruyter Expositions in Mathematics, No. 27, Walter de Gruyter, 1998.
9. M. D. M. Shaikh, S. Kumar, and S. K. Patra, *Recurrence in a dynamical system over adequate partial semigroups*.
10. N. Hindman, A. Maleki, D. Strauss, *Central sets and their combinatorial characterization*, J. Combin. Theory Ser. A, 74(2).