International Research Journal of Advanced Engineering and Science
ISSN (Online): 2455-9024

Unique Solution of Quadratic Integral Equation of Urysohn Type

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Abstract*— The paper is devote to unique solution of a nonlinear quadratic integral equation of Urysohn type via Banach contraction principle under some sufficient hypotheses and introduce example of it.*

Keyword*— quadratic integral equation, Banach contraction principle, semi norm, normed space, complete space.*

I. Introduction

Quadratic integral equation play a major role in the theory of radiative transfer, kinetic theory of gases, theory of neutron transport and traffic theory, see [2, 5, 6, 11], And especially, one of the most studied equations in nonlinear analysis is the Urysohn integral equation which it is considered useful in solving problems in real world, as well as, it has been treated by several authors as a paper [1, 8], It is worth mentioning, technique Urysohn integral equations by the Schauder theorem is not beneficial and the Banach contraction principle is very limited in lots of applications, so we need to discuss for equation of this type very carefully. There are many special case of Urysohn type for example [9, 10] which it Discussed functional Urysohn integral equation but the papers [4, 3], it Discussed functional quadratic integral equation of Urysohn type. Now we summarizes our work as follows:

- In the second part, we show all definitions, auxiliary Facts and theorems which will be needed in another parts of our paper.
- In the third part, we prove that the unique solution of quadratic integral equation of Urysohn type on interval $[0, M]$, using the Banach contraction principle.
- In the fourth part, it is the last and contains our conclusion.

II. Notation and Auxiliary Facts

In this section, we show that the definitions and theorems, as well the basic concept. Suppose Π is normed space with a family of semi norm $\mathcal{S}_{[x,y]}$ on bounded interval $[x, y]$, and suppose that $\mathcal{C}(\Pi)$ is all of real valued continues function which define by

$$\mathcal{S}_{[a,b]}(\psi) = \begin{cases} \sup_{a\le\tau\le b} |\psi(\tau)| & if\ \psi > 0 \\ 0 & if\ \psi = 0 \end{cases}$$

Such that $a, b \in \Pi$ and $a < b$ so, the last estimate satisfied the following

- $\mathcal{S}$ is symmetric.
- $\mathcal{S}(0) = 0$ and $\mathcal{S}$ is continuous at $\psi = 0$.
- $\mathcal{S}(a) \ge 0$ for all $a \in \Pi$.
- $\mathcal{S}$ is subadditive $i.e$ $\mathcal{S}(a + b) \le \mathcal{S}(a) + \mathcal{S}(b)$ for all $a, b \in \Pi$.
- $\mathcal{S}$ is positively homogeneous $i.e$ $\mathcal{S}(ra) = |r|\mathcal{S}(a)$ for all $a, b \in \Pi$ and $r \in \mathbb{R}$.

Definition 2.1 If $\mathcal{S}^{-1}(\{0\}) = \{0\}$ is satisfied then the semi norm $\mathcal{S}$ on Π is a norm.

Definition 2.2 [7] the function $\mathcal{S}: \Pi \to \Pi$, we can be say that it is a contraction mapping if all $n \in \mathcal{N}$ and there exists ℓ_n belong to (0,1) such that

$$\|\mathcal{S}(a) + \mathcal{S}(b)\| \le \ell_n \|a - b\| \qquad \forall\, a, b \in \Pi$$

Theorem 2.1 (Banach contraction principle) [7] Let the map $\mathcal{S}: \psi \to \psi$ is a contraction such that ψ is complete matric space than $\mathcal{S}$ has a unique fixed point in ψ.

III. Main Result

We can define the quadratic integral equation of Urysohn type as:

$$\psi(\tau) = \alpha(\tau) + \beta(\tau, \psi(\tau)) \int_0^K \gamma(\tau, \varsigma, \psi(\varsigma))d\varsigma \qquad \tau \in [0, M]$$

Suppose $\mathcal{S}: \mathcal{C}(\Pi) \to \mathcal{C}(\Pi)$ is operator defined by

$$\mathcal{S}\psi(\tau) = \alpha(\tau) + \beta(\tau, \psi(\tau)) \int_0^K \gamma(\tau, \varsigma, \psi(\varsigma))d\varsigma \qquad \tau \in [0, M] \qquad (3.1)$$

In this section, we will study the unique solution under the following hypothesis:

i) The function α by the form $\alpha: (0,1) \to \mathbb{R}$ is continuous.

ii) The function γ by the form $\gamma: I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a positive number $\mathcal{L}_n > 0$, for all $n \in \mathcal{N}$ then

$$|\gamma(\tau, \varsigma, \psi_1(\varsigma)) - \gamma(\tau, \varsigma, \psi_2(\varsigma))| \le \mathcal{L}_n |\psi_1 - \psi_2|$$

For each $\psi_1, \psi_2 \in \mathcal{C}(\Pi)$ and $\tau \in [0, M]$, $\varsigma \in [0, K]$.

But

$$|\gamma(\tau, \varsigma, \psi(\varsigma))| \le \omega(\varsigma)\phi(|\psi|)$$

Such that the function $\phi: I \to (0, \infty)$ is continuous and nondecreasing and $\omega \in \mathcal{C}(\Pi)$.

iii) The function β by the form $\beta: I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a positive number $\mathcal{L}_n^* > 0$, for all $n \in \mathcal{N}$ then

$$|\beta(\tau, \psi_1(\tau)) - \beta(\tau, \psi_2(\tau))| \le \mathcal{L}_n^* |\psi_1 - \psi_2|$$

For each $\psi_1, \psi_2 \in \mathcal{C}(\Pi)$ and $\tau \in [0, M]$.

But

Amany M. Moter and Shaymaa Shawkat Al-shakarchi, "Unique Solution of Quadratic Integral Equation of Urysohn Type," *International Research Journal of Advanced Engineering and Science*, Volume 9, Issue 1, pp. 155-157, 2024.

International Research Journal of Advanced Engineering and Science
ISSN (Online): 2455-9024

$$\left|\beta\left(\tau,\psi(\tau)\right)\right| \le \omega^*(\varsigma)\phi^*(|\psi|)$$

Such that the function $\phi^*: I \to (0,\infty)$ is continuous and nondecreasing and $\omega^* \in \mathcal{C}(\Pi)$.

iv) The following inequality is satisfy

$$K\omega^*(\varsigma)\phi^*\left(\mathcal{M}_{\mathcal{J}}\right)\mathcal{L}_n + K\omega(\varsigma)\phi\left(\mathcal{M}_{\mathcal{J}}\right)\mathcal{L}_n^* < 1$$

Theorem 3.1 If the assumptions $(i - iv)$ are satisfies then the equation (3.1) has unique solution.

Proof: First, we show that $\mathcal{C}(\Pi)$ is complete, so let $\mathcal{S}_n$ uniformly convergent sequence on Π such that

$$\lim_{n\to\infty}\|\mathcal{S}_n - \mathcal{S}\|_{\mathcal{J}} = 0,$$

Where $\mathcal{J} = [a,b]$, $i.e$ for any $\varepsilon > 0$, there exist an integer number m such that

$$\|\mathcal{S}_n - \mathcal{S}\|_{\mathcal{J}} < \frac{\varepsilon}{3} \qquad \forall m \ge n$$

Since $\mathcal{S}$ is continuous then there exist $c \in \Pi$, we can find positive real number δ such that

$d(a,c) < \delta$ that imply to $|\mathcal{S}_m(a) - \mathcal{S}_m(c)| < \frac{\varepsilon}{3}$

But, we obtain that

$$|\mathcal{S}(a) - \mathcal{S}(c)| \le |\mathcal{S}(a) - \mathcal{S}_m(a)| + |\mathcal{S}_m(a) - \mathcal{S}_m(c)| + |\mathcal{S}_m(c) - \mathcal{S}_m(c)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

We conclude that the space $\mathcal{C}(\Pi)$ is a complete.

Next, let the set Ψ is bounded and closed subset of $\mathcal{C}(\Pi)$,defined by $\Psi = \{\psi \in \mathcal{C}(\Pi): \|\psi\|_{\mathcal{J}} \le \mathcal{M}_{\mathcal{J}},\ \mathcal{J} = [a,b]\ \}$, so we shall prove that $\mathcal{A}$: $\Psi \to \mathcal{C}(\Pi)$ is contraction operator. As well, suppose $\psi_1, \psi_2 \in \Psi$ for all $\tau \in [0,M]$.

$$|\mathcal{S}\psi_1(\tau) - \mathcal{S}\psi_2(\tau)| = \Big|\alpha(\tau) + \beta\left(\tau,\psi_1(\tau)\right)\int_0^K \gamma(\tau,\varsigma,\psi_1(\varsigma))d\varsigma - \alpha(\tau) - \beta\left(\tau,\psi_1(\tau)\right)\int_0^K \gamma(\tau,\varsigma,\psi_2(\varsigma))d\varsigma + \alpha(\tau) + \beta\left(\tau,\psi_1(\tau)\right)\int_0^K \gamma(\tau,\varsigma,\psi_2(\varsigma))d\varsigma - \alpha(\tau) - \beta\left(\tau,\psi_2(\tau)\right)\int_0^K \gamma(\tau,\varsigma,\psi_2(\varsigma))d\varsigma\Big|$$

Implying that

$$|\mathcal{S}\psi_1(\tau) - \mathcal{S}\psi_2(\tau)| = \Big|\beta\left(\tau,\psi_1(\tau)\right)\int_0^K [\gamma\left(\tau,\varsigma,\psi_1(\varsigma)\right) - \gamma\left(\tau,\varsigma,\psi_2(\varsigma)\right)]d\varsigma + [\beta\left(\tau,\psi_1(\tau)\right) - \beta\left(\tau,\psi_2(\tau)\right)]\int_0^K \gamma(\tau,\varsigma,\psi_2(\varsigma))d\varsigma\Big|$$

So, we obtain

$$|\mathcal{S}\psi_1(\tau) - \mathcal{S}\psi_2(\tau)| \le \left|\beta\left(\tau,\psi_1(\tau)\right)\right|\int_0^K \left|\left[\gamma\left(\tau,\varsigma,\psi_1(\varsigma)\right) - \gamma\left(\tau,\varsigma,\psi_2(\varsigma)\right)\right]\right|d\varsigma + \left|\beta\left(\tau,\psi_1(\tau)\right) - \beta\left(\tau,\psi_2(\tau)\right)\right|\int_0^K |\gamma(\tau,\varsigma,\psi_2(\varsigma))|d\varsigma$$

By assumption $(ii - iv)$, we have

$$|\mathcal{S}\psi_1(\tau) - \mathcal{S}\psi_2(\tau)| \le K\omega^*(\varsigma)\phi^*(|\psi_1|)\mathcal{L}_n|\psi_1 - \psi_2| + K\omega(\varsigma)\phi(|\psi_2|)\mathcal{L}_n^*|\psi_1 - \psi_2|$$
$$\le (K\omega^*(\varsigma)\phi^*(|\psi_1|)\mathcal{L}_n + K\omega(\varsigma)\phi(|\psi_2|)\mathcal{L}_n^*)|\psi_1 - \psi_2|$$

Hence

$$\|\mathcal{S}\psi_1(\tau) - \mathcal{S}\psi_2(\tau)\|_{\mathcal{J}} \le (K\omega^*(\varsigma)\phi^*\left(\mathcal{M}_{\mathcal{J}}\right)\mathcal{L}_n + K\omega(\varsigma)\phi\left(\mathcal{M}_{\mathcal{J}}\right)\mathcal{L}_n^*)\|\psi_1 - \psi_2\|_{\mathcal{J}}$$

$\mathcal{S}$ is a contraction, so we conclude that the operator $\mathcal{S}$ has solution ψ on Ψ.

Example 3.1. Suppose the quadratic differential equation Urysohn type:

$$\mathcal{S}\psi(\tau) = ln\tau + \frac{\psi(\tau)}{\tau^2 + \varsigma + 2}\int_0^N (5 - \tau^{\varsigma})\psi(\varsigma)d\varsigma \qquad \tau \in [0,M]$$

By comparison, the above equation and equation (3.1) we have:

$ln\tau = \alpha(\tau)$, $\quad \tau \in I$

$\beta\left(\tau,\psi(\tau)\right) = \frac{\psi(\tau)}{\tau^2+2}$ $\quad \tau \in I$ and $\psi \in \mathcal{C}(\Pi)$

$\gamma(\tau,\varsigma,\psi(\varsigma)) = (5 - \tau^{\varsigma})\psi(\varsigma)$ $\quad \forall\ \psi_1,\psi_2 \in \mathcal{C}(\Pi)$ and $\tau \in [0,M]$, $\varsigma \in [0,K]$

All that remains is to check the hypothesis $(i - iv)$, thus the function $ln\tau$ is continuous implying that the hypothesis (i) is satisfies, next

$$\left|\beta\left(\tau,\psi_1(\tau)\right) - \beta\left(\tau,\psi_2(\tau)\right)\right| = \left|\frac{\psi_1(\tau)}{\tau^2 + \varsigma + 2} - \frac{\psi_2(\tau)}{\tau^2 + \varsigma + 2}\right| = \frac{1}{\tau^2 + \varsigma + 2}|\psi_1(\tau) - \psi_2(\tau)|$$

We have

$$\left|\beta\left(\tau,\psi_1(\tau)\right) - \beta\left(\tau,\psi_2(\tau)\right)\right| \le \frac{1}{2}|\psi_1(\tau) - \psi_2(\tau)|$$

And $\left|\beta\left(\tau,\psi(\tau)\right)\right| = \left|\frac{\psi(\tau)}{\tau^2+\varsigma+2}\right| \le \frac{1}{\tau^2+\varsigma+2}|\psi(\tau)| \le \frac{1}{\varsigma+2}|\psi(\tau)|$

We get that the hypothesis (ii) is satisfies with $\mathcal{L}_n^* = \frac{1}{2}$ and $\omega^*(\varsigma) = \frac{1}{\varsigma+2}$, for all $\psi \in \mathcal{C}(\Pi)$, $\tau \in I$ and $\varsigma \in [0,K]$.

Finally,

$$\left|\gamma\left(\tau,\varsigma,\psi_1(\varsigma)\right) - \gamma\left(\tau,\varsigma,\psi_2(\varsigma)\right)\right| = \left|\left(5 - \frac{1}{\varsigma} + \psi_1\tau\right)\psi_1(\varsigma) - \left(5 - \frac{1}{\varsigma} + \psi_2\tau\right)\psi_2(\varsigma)\right|$$

We have

$$\left|\gamma\left(\tau,\varsigma,\psi_1(\varsigma)\right) - \gamma\left(\tau,\varsigma,\psi_2(\varsigma)\right)\right| \le 5|\psi_1 - \psi_2|$$

And

$$\left|\gamma\left(\tau,\varsigma,\psi(\varsigma)\right)\right| = \left|\left(5 - \frac{1}{\varsigma} + \psi\tau\right)\psi(\varsigma)\right| \le \left(5 - \frac{1}{\varsigma}\right)|\psi(\varsigma)|$$

We get that the hypothesis (iii) is satisfies with $\mathcal{L}_n = \frac{1}{2}$ and $\omega(\varsigma) = 5 - \frac{1}{\varsigma}$, for all $\psi \in \mathcal{C}(\Pi)$, $\tau \in [0,M]$ and $\varsigma \in [0,K]$.

IV. CONCLUSION

We could prove that the Urysohn equation (3.1) has unique solution by use a technique related with Banach contraction principle in $\mathcal{C}(\Pi)$ by four assumptions. Finally, there was an example to proving accuracy solution.

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