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Innovative Practice and Teaching Research of Functional Analysis in Local Undergraduate Colleges

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Abstract*—*Local undergraduate students have relatively weak foundations, making it difficult to learn the course of functional analysis. Therefore, it is necessary to explore teaching methods related to functional analysis to help students learn this course well. In the teaching process, compare with the concept and theory of finite dimensional spaces, connect with related courses such as linear algebra, geometry, topology, etc., Apply basic theories to practical mathematical problems to stimulate students' interest in learning, and be good at using counterexamples to understand the learned theory.

Keywords*—*Application, Teaching, Functional analysis.

I. INTRODUCTION

Functional analysis is a mathematical discipline developed in the 1930s, with quantum mechanics as its main application background. It uses function theory, geometry, and other methods to study infinite dimensional spaces and operator theory on them. Due to the relatively weak foundation of local undergraduate students, it is difficult to learn the very difficult subject-functional analysis. Therefore, it is necessary to explore teaching methods related to functional analysis to help students learn this course well.

II. COMPARE WITH FINITE DIMENSIONAL SPACE

Functional analysis studies the algebraic and topological structures of infinite dimensional spaces. There are many differences and connections in related theories between finite dimensional spaces and infinite dimensional spaces. In the teaching process, attention should be paid to guiding students to learn to think and compare with them.

The algebraic structure of two types of spaces can also be called a linear structure, which refers to the two linear operations of addition and multiplication defined on a non empty set, and the two operations are closed, thus becoming a linear space. Researching whether the linear structure between two linear spaces is the same, usually examining whether they are isomorphic, that is, whether there exists a one-to-one mapping that preserves the closure of addition and multiplication operations. The necessary and sufficient condition that two finite dimensional linear spaces are isomorphic is that their dimensions are the same. Therefore, studying the algebraic structure of finite dimensional spaces only requires mastering the algebraic structure of n-dimensional Euclidean spaces.

The topological structures of the two types of spaces are different, and the important difference is that the unit sphere of an infinite dimensional space is no longer column compact, while the unit sphere of a finite dimensional space is column compact. Therefore, a bounded infinite point sequence in an infinite dimensional space may not necessarily have a convergent subsequence, such as a point sequence

$$x_i = (0,\cdots,0,1,0,0,\cdots), i = 1,2,3,\cdots$$

where the i-th component is 1, and all others are 0. This infinite sequence obviously has no convergent subsequence, but for finite dimensional spaces, any bounded infinite point sequence must have a convergent subsequence.

In the teaching process, it is important to emphasize this important difference and provide examples to illustrate the difference between the two tyes of spaces, so that students can understand that the transition from finite dimensional space to infinite dimensional space is not a simple extension, but a qualitative change.

III. COMPARE WITH SOME LEARNED COURSES

Connect abstract spatial concept and properties with learned courses such as linear algebra, topology, geometry, etc., in order to deepen the understanding and consolidation of new and old knowledge.

A metric space is a special type of topological space that studies the topological structure of a space by constructing a metric structure on a non-empty set. That is, defining a bivariate real-valued function that satisfies non-negativity, symmetry, and trigonometric inequalities. It is fundamentally different from linear spaces. For example, a set $[0,1]$ can be defined using the usual metric formula to form a metric space, but it cannot form a linear space using the usual addition and multiplication operations. In addition, different metric structures can be defined on the same non-empty set to form different metric spaces. For example, metrics d, d_1, d_2 can be defined on $C[a,b]$, which are

$$d(x,y) = \max_{a \le t \le b} |x(t) - y(t)|,$$

$$d_1(x,y) = \int_a^b |x(t) - y(t)| dt,$$

$$d_2(x,y) = \left(\int_a^b |x(t) - y(t)|^2 dt\right)^{\frac{1}{2}}$$

for any $x, y \in C[a,b]$.

A normed linear space contains both the topological structure and algebraic structure of a space. From an algebraic perspective, it is a linear space and from a topological perspective, it is a special topological space. It refers to defining a norm structure on the basis of a linear space, thus

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becomes a normed linear space, such as $(X,\|\cdot\|)$. The metric can be obtained from the norm

$$d(x,y)=\|x-y\|, x,y\in X,$$

which is called a metric induced by norm, thus becomes a metric space. Conversely, the metric structure of a metric space may not necessarily be generated by a certain norm, because metric induced by norm has the translational invariance of metric and the continuity of multiplication. These differences and connections must be compared and explained during the teaching process, otherwise students can easily cause confusion.

An inner product space is a linear space that is equipped with an inner product structure, and its inner product structure can induce a norm

$$\|x\|=(x,x)^{\frac{1}{2}},$$

which is called the norm induced by inner product and satisfies the parallelogram rule. So, each inner product space can generate a normed linear space, and the norm structure of each normed linear space may not necessarily be generated by a cretain inner product. With the inner product structure, the angle between two elements can be defined, which is organically related to the angle between two vectors in three-dimensional space. With geometric significance, the concept of functional analysis becomes relatively simple.

If the inner product of two elements in R^n is

$$(x,y)=\sum_{i=1}^{n}x_i y_i,$$

where

$$x=(x_1,x_2,\cdots,x_n), y=(y_1,y_2,\cdots,y_n)\in R^n.$$

In addition $(x,y)=\|x\|\cdot\|y\|\cdot\cos\alpha$, where α is the angle between two vectors. It is related to the quantity product

$$x\cdot y=|x||y|\cos\alpha$$

of two vectors in geometry. Therefore, it can be said that the inner product of the two elements in R^n is an extension of the concept of quantity product. Therefore, some properties of quantity product can be naturally extended to inner product.

When the inner product of two non-zero elements is 0, it means that the angle between these two vectors is $\frac{\pi}{2}$, that is, these two vectors are perpendicular to each other, which is called that the two elements are orthogonal. Orthogonal non-zero vectors are linearly independent, which connects the relevant theories of functional analysis with courses such as geometry and advanced algebra, and helps students learn and master these theories better.

IV. Connect With Some Practical Problems

We can emphasize the connection between theory and practice, apply complex and abstract concepts and theories to practical problems, and stimulate students' interest in learning.

The theoretical abstraction of functional analysis is difficult to understand, and students find it dull and uninteresting to learn. If abstract theories can be applied concretely and their practical significance can be realized, it can deepen their understanding of the theory. When studying the Banach fixed point theorem, we point out that applying the Banach fixed point theorem we can prove the solution of the initial value problem of first-order differential equations

$$\begin{cases}\frac{dx}{dt}=f(x,t)\\ x(0)=x_0\end{cases}$$

exists and is unique when certain conditions are met. This fully demonstrates the important value of functional analysis theory in studying many problems of ordinary differential equations, expands students' horizons, and stimulates their interest in learning.

The abstract Arzela-Ascoli theorem plays an important role in studying whether a set is column-compact or not. In mathematical analysis, if

$$A=\{x(t)\in C^1[a,b]:|x(t)|\le M,|x'(t)|\le M\}$$

then A is a column-compact set of C [a, b]. When it comes to standard orthogonal bases, a standard orthogonal basis on $L^2[0,2\pi]$ can be given as

$$\{\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{\pi}}\cos t,\frac{1}{\sqrt{\pi}}\sin t,\frac{1}{\sqrt{\pi}}\cos 2t,\frac{1}{\sqrt{\pi}}\sin 2t,\cdots,\frac{1}{\sqrt{\pi}}\cos nt,\frac{1}{\sqrt{\pi}}\sin nt,\cdots\}$$

Any element in space can be written as a linear combination of this set

$$x(t)=(x,\frac{1}{\sqrt{2\pi}})\cdot\frac{1}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}[(x,\frac{1}{\sqrt{\pi}}\cos nt)\cdot\frac{1}{\sqrt{\pi}}\cos nt+(x,\frac{1}{\sqrt{\pi}}\sin nt)\cdot\frac{1}{\sqrt{\pi}}\sin nt]$$

where $(x,y)=\int_0^{2\pi}x(t)\overline{y(t)}dt$. This happens to be the Fourier series in mathematical analysis, which makes students not only learn new concepts but also deepen the understanding of the theory of mathematical analysis.

After learning the concepts of unbounded operators and spectra, it can be pointed out that an operator are actually a mapping or a transformation. Linear transformations and symmetric transformations learned in linear algebra are special bounded linear operators, while many abstract operators in practical problems are unbounded, such as differential operators

$$-y''(x)=\lambda y(x), x(0)=x(1)=0$$

which is the boundary value problem of the Schrodinger equation with an infinite depth square potential well in quantum mechanics. In addition, the concept of spectrum is actually an extension of the concept of eigenvalues of matrix in linear algebra. For the above differential operators, if there is a real number that causes the boundary value problem of the differential equation has non-trivial solutions, then this real number is called a point spectrum of the operator, which is the eigenvalue of the problem. After learning self-adjoint operators on Hilbert spaces and connecting with symmetric transformations in higher algebra, it is pointed out that symmetric transformations are actually bounded self-adjoint operators. The matrix of symmetric transformations on a set of standard orthogonal bases is a symmetric matrix, and the eigenvalues of symmetric matrices are all real-valued. The eigenvectors corresponding to different eigenvalues are

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orthogonal. These properties of symmetric matrices can be extended to self-adjoint operators, which can make students understand the concept and properties of self-adjoint operators better. This closely links basic concepts, theories with classical quantum mechanics problems, and connects newly learned concepts and theories with previously learned concepts, which helps students understand the relationship between the knowledge they have learned.

V. SKILLED IN USING COUNTEREXAMPLES

In the teaching process of functional analysis, being good at using counterexamples to illustrate problems often leads to better teaching effectiveness. Define a metric

$$\rho(x,y)=\max_{t\in[0,1]}\left|x(t)-y(t)\right|,$$

in a metric space $C[0,1]$, which creates a complete metric space. But if a metric

$$\rho_1(x,y)=\int_0^1\left|x(t)-y(t)\right|dt$$

is defined in this space, then the space is incomplete, because the Cauchy column

$$x_n(t)=\begin{cases}0, & 0\le t\le\frac{1}{2}-\frac{1}{n}\\ 1, & \frac{1}{2}\le t\le 1\\ a\quad line, & otherwise\end{cases}$$

of the space does not converge.

A normed linear space is always a metric space, and conversely, metrics in a metric space may not necessarily be generated by a certain norm. In this case, an example can be used to illustrate. Let space $S=\{\{\xi_n\}\}$ be the set of all real columns, and define the distance on S

$$\rho(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\left|\xi_k-\eta_k\right|}{1+\left|\xi_k-\eta_k\right|}$$

where

$$x=(\xi_1,\xi_2,\cdots,\xi_k,\cdots),\ y=(\eta_1,\eta_2,\cdots,\eta_k,\cdots)\in S$$

S is a complete metric space, but it does not satisfy the continuity of multiplication, and therefore cannot be generated by a certain norm. When it comes to norms in normed linear spaces, they may not necessarily be generated by inner products. A counterexample can be used to illustrate this, such as

$$(C[0,1],\|\cdot\|),\|x\|=\max_{t\in[0,1]}\left|x(t)\right|.$$

At this point, the norm does not satisfy the parallelogram rule and therefore cannot be generated by any inner product.

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