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Natural Frequencies and Mode Shapes for Vibrations of Rectangular and Circular Membranes: A Numerical Study

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Abstract— *The main goal of present study is to simulate vibrations of rectangular and circular membranes using COMSOL software. In the simulation procedure, two rectangular and one circular membranes with constant thickness have been modeled in COMSOL software. For rectangular and circular membranes, external edges along the perimeter considered to be fixed at zero displacement to simulate the boundary conditions as the Dirichlet boundary conditions. Then natural frequencies and mode shapes are obtained by considering an initial displacement, for simulated membranes. In order to validate the results, two-dimensional wave equation solved for rectangular membranes by considering the Dirichlet boundary conditions. Natural frequencies are derived from the wave equation analytically and are consistent with COMSOL simulations with less than 1% difference*.

Keywords— *Natural Frequencies, Vibrations of Membranes, COMSOL software*.

I. INTRODUCTION

Engineers and scientists have been trying to solve the constitutive equations which they have derived for their models. In many cases, the constitutive equations include linear or nonlinear partial differential equation (PDEs). [1-14] Solving the linear and nonlinear PDEs is not usually an easy solution and nonlinear coefficients make the PDEs very complicated. COMSOL is a useful cross-platform finite element analysis software designed for different simulation problems such as heat transfer, stress analysis, vibration and so on. [14-22]. ABAQUS software is one of the popular FEM software which has been used for wide range of study. Extracting accurate results in ABAQUS depend on defining the boundary conditions, steps of the solution, type, and size of meshes carefully [22-34]. Handful heat transfer, stress analysis and vibrations studied have been done using available commercial FEM software. Currently, COMSOL is using widely because not only is a user-friendly software but also allows coupling a linear or nonlinear equation to the model. [34-50]

The objective of the present study is to simulate the vibrations of rectangular and circular membranes by using COMSOL software. In the simulation procedure, two rectangular and one circular membranes have been modeled for extracting the natural frequencies and mode shapes during vibrations. Dimensions of the first and second rectangles are 4 by 2 ft. and 4 by 3 ft. respectively. To defining the Dirichlet boundary condition, external edges along the perimeter considered to be fixed at zero displacement. Then response of each rectangular membrane subjected to a given initial displacement condition obtained. For validation purposes, two-dimensional wave equation solved for rectangular membranes by considering the Dirichlet boundary conditions. Natural Frequencies obtained by the wave equation and COMSOL software have less than 1% discrepancy.

II. VIBRATIONS OF THE 2D RECTANGULAR AND CIRCULAR MEMBRANES

For a rectangular membrane in the x-y plane with a length of L and width of K the wave equation can be written as:

$$\frac{\partial^2 z}{\partial t^2} = C^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

To describe the Dirichlet boundary conditions, external edges along the perimeter assumed to be fixed at zero displacement. Moreover, the initial velocity is zero at the onset of the vibration. Boundary conditions for vibrations of a rectangular membrane can be written as:

$$Z(x,0,t) = 0$$
$$Z(x,k,t) = 0$$
$$Z(0,y,t) = 0$$
$$Z(L,y,t) = 0$$
$$\frac{\partial Z}{\partial t}(x,y,0) = 0$$

The solution of the wave equation for the selected boundary conditions is as:

$$Z(x,y,t) = \sum_{n=1}^{\alpha}\sum_{m=1}^{\alpha} a_{nm}.\sin\frac{n\prod x}{L}\sin\frac{m\prod y}{k}\cos(w_{nm}t)$$

where, w_{nm} is the natural frequency and can be written as:

$$w_{nm} = \prod C.\sqrt{\frac{n^2}{L^2} + \frac{m^2}{K^2}}$$

For a circular membrane in the x-y plane with a radius of R, the wave equation can be written in polar coordinates as:

$$\frac{\partial^2 z}{\partial t^2} = C^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}.\frac{\partial z}{\partial \theta} + \frac{1}{r}.\frac{\partial^2 z}{\partial \theta^2}\right)$$

The Dirichlet boundary conditions are defined in a way that external edges along the perimeter of the circle are fixed at zero displacement similar to rectangular membranes. Likewise, the initial velocity presumed to be zero at t=0. The

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boundary conditions for vibrations of a circular membrane can be written as:

$$Z(R,t)=0$$

$$\frac{\partial Z}{\partial t}(r,0)=0$$

The solution to the wave equation for the selected boundary conditions is as:

$$Z(r,t)=\sum_{n=1}^{\alpha} a_n . J_0\left(\frac{J_n r}{R}\right).\cos\left(\frac{J_n C}{R}t\right)$$

III. Results of Natural Frequencies and Mode Shapes for Vibrations of Rectangular and Circular Membranes

In this study, two rectangular and one circular membranes have been modeled for extracting the natural frequencies and mode shapes under vibrations. Dimensions of the first rectangle selected as 4 by 2 ft. Wave equation and boundary condition for the first rectangular membrane is as follows:

$$\frac{\partial^2 Z}{\partial t^2}=25.\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

$$Z(x,y,0)=0.1(4x-x^2).(2y-y^2)\ )$$

$$\frac{\partial Z}{\partial t}(x,y,0)=0$$

Figure 1 shows natural frequencies and mode shapes for vibrations of the first rectangular membrane.

Table 1 demonstrates a comparison between the analytical solution of the PDE and COMSOL results for the natural frequencies of the first rectangular membrane.

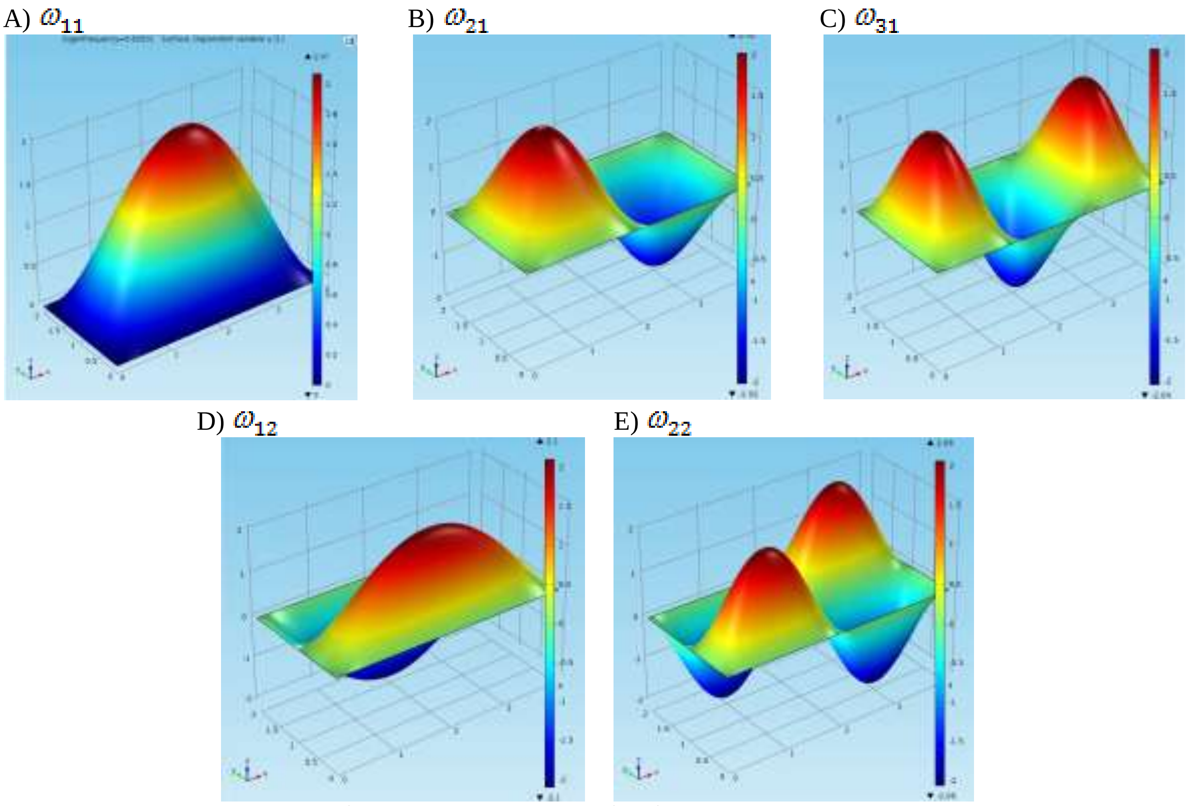


Fig. 1. Natural frequencies and mode shapes of the first vibrating rectangular membrane.

TABLE 1. Comparison between the analytical solution of the PDE and COMSOL results

NATURAL FREQUENCIES	COMSOL	PDE	DIFFERENCE (%)
ω_{11}	1.25002π	1.25 π	0.0016
ω_{21}	1.5812 π	1.5811 π	0.0063
ω_{31}	2.0158 π	2.01556 π	0.0119
ω_{12}	2.305 π	2.3048 π	0.0087
ω_{22}	2.5004 π	2.5 π	0.0160

Dimensions of the second rectangle are 4 by 3 ft. Wave equation and boundary condition for the second rectangular membrane is as follow:

$$\frac{\partial^2 Z}{\partial t^2}=8.\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

$$Z(x,y,0)=0.1(4x-x^2).(3y-y^2)$$

$$\frac{\partial Z}{\partial t}(x,y,0)=0$$

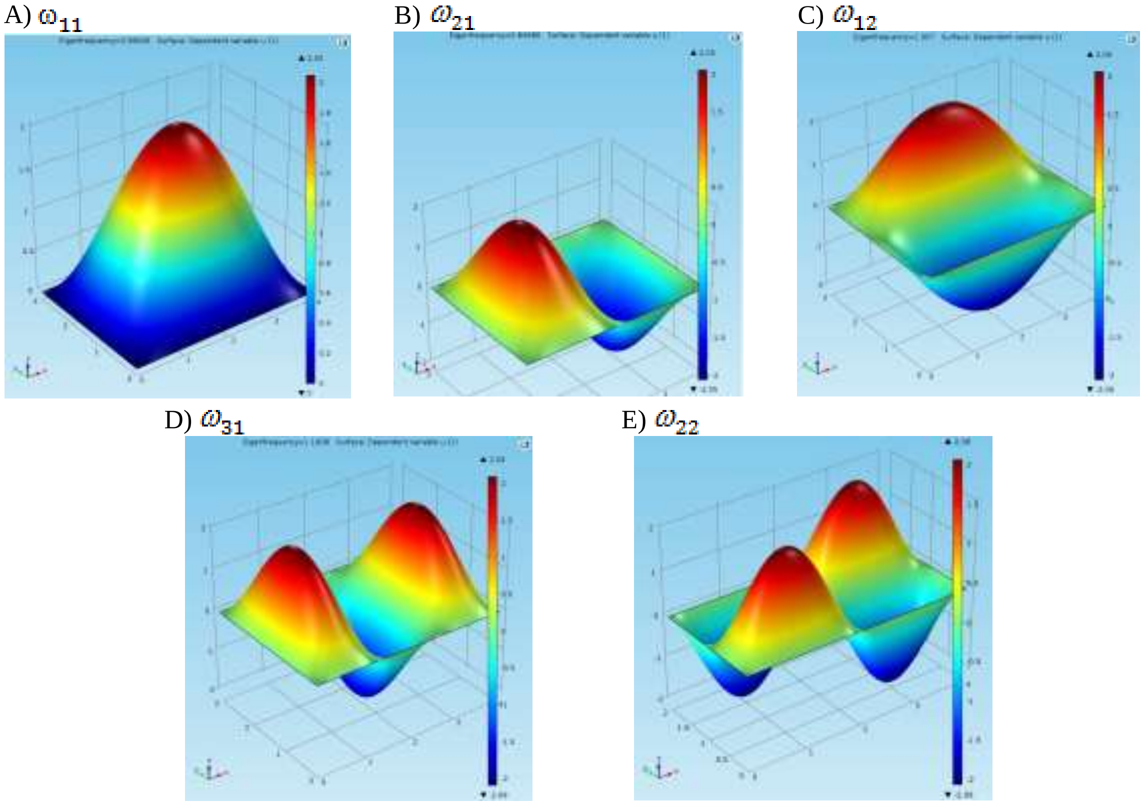


Fig. 2. Natural frequencies and mode shapes of the second vibrating rectangular membrane.

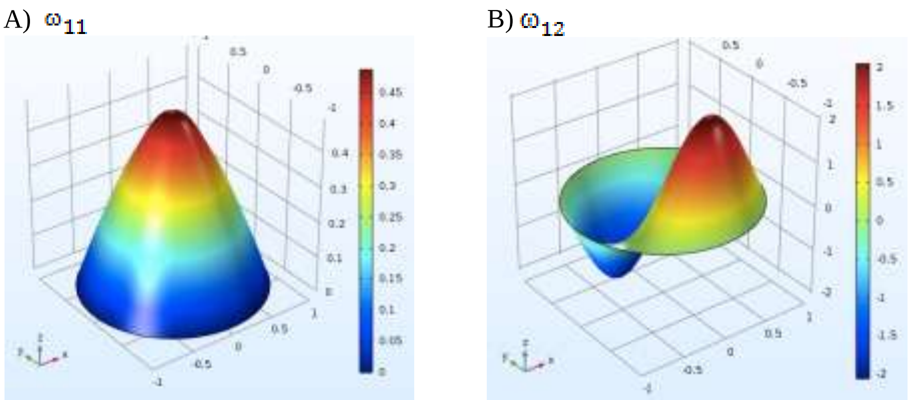
Figure 2 shows natural frequencies and mode shapes for vibrations of the second rectangular membrane.

Table 2 illustrates a comparison between the analytical solution of the PDE and COMSOL results for natural frequencies of the second rectangular membrane.

Finally, as it can be seen, Figure 3 shows the results of studying the vibrations of circular membranes to determine frequencies and mode shapes.

TABLE 2. the Analytical and simulation results of the second vibrating rectangular membrane.

NATURAL FREQUENCIES	COMSOL	PDE	DIFFERENCE (%)
ω_{11}	1.17852 π	1.17833 π	0.0161
ω_{21}	1.69972 π	1.69941 π	0.0182
ω_{31}	2.014 π	2.01356 π	0.0218
ω_{12}	2.3216 π	2.321 π	0.0258
ω_{22}	2.3572 π	2.3566 π	0.0255



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C) ω_{21} D) ω_{22}

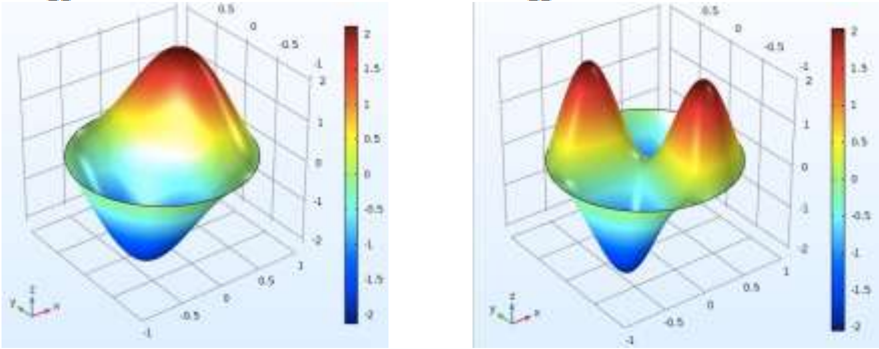

Fig. 3. Natural frequencies and mode shapes for vibration of the circular membrane.

IV. Conclusion

COMSOL software is a user-friendly finite element analysis package that can be used to solve linear or nonlinear partial differential equations (PDEs) in different areas such as heat transfer, stress analysis, and vibrations. In the present study, two rectangular and one circular membranes are modeled in order to extract the natural frequencies and mode shapes during the vibrations. For validation purposes, two-dimensional wave equation solved for rectangular membranes by considering the Dirichlett boundary conditions. Natural Frequencies obtained from the wave equations and COMSOL are in a very good agreement with less than 1% difference that shows the accuracy of COMSOL solutions.

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