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Results in Generalized Hyperbolic Functions

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Abstract*—*This paper considers the study of generalized fractional hyperbolic like functions. These functions are obtained as solutions of a specific type of 3rd order and nth order CFDEs involving a parameter 'a'. Inequality results are obtained relating the solutions of the considered CFDEs.

Keywords*— *Caputo fractional differential equation, generalized fractional hyperbolic functions, Wronskian.

***Mathematics Subject Classification**: Primary: 47G20.*

I. Introduction

In [7], using the technique in [6] generalized fractional hyperbolic functions are obtained as solutions of *2nd* order CFDE involving a parameter '*a*'. In this paper we consider *3rd* order and higher order CFDEs of the same family and obtain their solutions using the approach in [7]. The properties of these solutions are also studied.

II. Preliminaries

To obtain the main results in this paper we need to introduce definitions and concepts related to fractional derivatives. These definitions run parallel to the definitions of ordinary derivatives. In this context we first begin with a generalization of the exponential function known as Mittag - Leffler function which was discovered in 1903 [4, 8].

Definition 2.1: The Mittag - Leffler function of one parameter, $E_q(z)$ is defined by

$$E_q(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+1)} \quad (z\in C,\ R(q)>0)\,. \tag{2.1}$$

where the symbol Γ denotes Gamma function.

Definition 2.2: The Mittag - Leffler function of two parameters, $E_{q,\beta}(z)$ is defined by

$$E_{q,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+\beta)} \quad (z,\beta\in C,\ R(q)>0)\,. \tag{2.2}$$

The definitions of fractional derivatives for a series introduced by Riemann and Caputo [3] are given below.

Definition 2.3: Riemann - Liouville fractional derivative for series.
If

$$f(x)=x^{q-1}\sum_{k=0}^{\infty}a_k x^{kq}$$

then

$$D^q f(x)=\frac{d^q(f(x))}{dx^q}=x^{q-1}\sum_{k=0}^{\infty}a_{k+1}\frac{\Gamma((k+2)q)}{\Gamma((k+1)q)}x^{kq} \tag{2.3}$$

Definition 2.4: Caputo fractional derivative for series.
If

$$f(x)=\sum_{k=0}^{\infty}a_k x^{kq}$$

then

$${}^cD^q f(x)=\frac{d^q(f(x))}{dx^q}=\sum_{k=0}^{\infty}a_{k+1}\frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)}x^{kq} \tag{2.4}$$

Next we proceed to present the definitions of the fore mentioned derivatives in terms of the integrals.

Definition 2.5: Riemann - Liouville derivative of $x(t)$ is given by

$$D^q x(t)=\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{t_0}^{t}(t-s)^{-q}x(s)ds,\quad (t\in\square). \tag{2.5}$$

Definition 2.6: Caputo derivative of x(t) is given by

$${}^cD^q x(t)=\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x'(s)ds,\quad (t\in\square) \tag{2.6}$$

The initial value problem for Riemann - Liouville fractional differential equation (RLFDE) and the initial value problem for Caputo fractional differential equation (CFDE) have a basic difference. The RLFDE has a singularity at the initial point and is given by

$$D^q x(t)=f(t,x(t))\ ,\ x^0=x(t)(t-t_0)^{1-q}/t=t_0,$$

and the CFDE is given by

$${}^cD^q x(t)=f(t,x(t))\ ,\ x(t_0)=x_0.$$

There exists a relation between the CFDE and RLFDE which is given by

$${}^cD^q x(t)=D^q[x(t)-x_0].$$

It has been shown in [2, 5] that the results which hold for the initial value problem of RLFDE are also true for CFDE. On basis of this result we give the existence and uniqueness results for linear n^{th} order RLFDE and for systems and propose that they can be naturally extended for linear CFDE. We now introduce the q - exponential function which is needed to define the solution of the linear Reimann - Liouville fractional differential equation.

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Definition 2.7: The q - exponential function $e_q^{\lambda z}$ is defined by

$$e_q^{\lambda z} = z^{q-1}E_{q,q}(\lambda z^q) \tag{2.7}$$

where $(z \in C\setminus\{0\}, R(q)>0)$ and $\lambda \in C.$

Definition 2.8: We define the function $e_{q,n}^{\lambda z}$ as

$$e_{q,n}^{\lambda z} = z^{q-1}\sum_{k=0}^{\infty}\frac{(k+n)!}{\Gamma[(k+n+1)q]}\frac{(\lambda z^q)^k}{k!}. \tag{2.8}$$

Consider the linear fractional differential equation (LFDE).

$$[L_{nq}(y)](t) := (D_{a^+}^{nq})y(t) + \sum_{k=0}^{n-1}a_k(D_{a^+}^{kq})y(t) = 0 \tag{2.9}$$

where the coefficients $\{a_j\}_{j=1}^{n-1}$ are real constants.

Then we assume that the solution of (2.9) is of the form

$$y(t) = e_q^{\lambda(t-a)}, \lambda \in C$$

and obtain the characteristic equation as

$$P_n(\lambda) = \lambda^n + \sum_{k=1}^{n-1}a_k\lambda^k, \lambda \in C. \tag{2.10}$$

Please refer to [5] for lemmas and theorems that are necessary to obtain the existence and uniqueness result for LFDE (2.9).

We denote $\mathbb{R}^+$ as the set of all non-negative real numbers.

Parallel to the definition of Wronskian in ordinary differential equations [2] we define Wronskian corresponding to fractional differential equations as follows:

Definition 2.9: (Wronskian).

Let $\phi_1,\phi_2,\ldots,\phi_n$ be n real or complex valued functions defined on some nonempty interval I in $\mathbb{R}_+$ each having derivatives of order $\alpha = nq,\ n\in N.$ Then the fractional Wronskian of these n functions is the determinant matrix of the W of order n defined on I and whose value at $t \in I$ is

$$W(t) = W(\phi_1,\phi_2,\phi_3,...,\phi_n)(t)$$

$$= \begin{vmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ {}^cD^q\phi_1(t) & {}^cD^q\phi_2(t) & \cdots & {}^cD^q\phi_n(t) \\ \vdots & \vdots & \vdots & \vdots \\ {}^cD^{(n-1)q}\phi_1(t) & {}^cD^{(n-1)q}\phi_2(t) & \cdots & {}^cD^{(n-1)q}\phi_n(t) \end{vmatrix} \tag{2.11}$$

III. Generalized Fractional Hyperbolic Like Functions Through Third Order CFDE

In [8] many results pertaining to 2nd order CFDE are stated and proved. In this section we state and prove important results corresponding to 3rd order CFDE using the theory of fractional differential equations.

We now state and prove a theorem in which generalized fractional hyperbolic like functions are obtained.

Consider the $(3q)^{th}$ order, $\left(\frac{2}{3}<q\le 1\right)$ homogeneous Caputo fractional IVP

$${}^cD^{3q}x(t) - a^q x(t) = 0, \tag{3.1}$$

$$x(0)=1,\ {}^cD^q x(0)=0,\ {}^cD^{2q}x(0)=0 \tag{3.2}$$

where $t\in\mathbb{R}_+,\ a>0$ is a real number.

Theorem 3.1. The general solution of the CFDE (3.1) is given by $c_1x(t)+c_2y(t)+c_3z(t)$ (c_1, c_2 and c_3 being arbitrary constants) where $x(t)$, $y(t)$ and $z(t)$ are infinite series solutions of the form

$$x(t) = \sum_{k=0}^{\infty}\frac{(\sqrt[3]{at})^{3kq}}{\Gamma(1+3kq)}, \tag{3.3}$$

$$y(t) = \sum_{k=0}^{\infty}\frac{(\sqrt[3]{at})^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \tag{3.4}$$

$$z(t) = \sum_{k=0}^{\infty}\frac{(\sqrt[3]{at})^{(3k+2)q}}{\Gamma(1+(3k+2)q)},\quad t\in\mathbb{R}_+. \tag{3.5}$$

Proof. We transform the given IVP to a system of equations of q^{th} order $\frac{2}{3}<q\le 1$ as

$$\begin{aligned} &{}^cD^q x(t) = (\sqrt[3]{a})^q z(t), \\ &{}^cD^q y(t) = (\sqrt[3]{a})^q x(t),\ \ {}^cD^q z(t) = (\sqrt[3]{a})^q y(t) \end{aligned} \tag{3.6}$$

with initial conditions

$$x(0)=1,\ y(0)=0,\ z(0)=0. \tag{3.7}$$

Let

$$x(t)=\sum_{k=0}^{\infty}a_k t^{kq},\ \ y(t)=\sum_{k=0}^{\infty}b_k t^{kq},\ \ z(t)=\sum_{k=0}^{\infty}c_k t^{kq} \tag{3.8}$$

be solutions of the system (3.6) - (3.7) where a_k , b_k and $c_k{}'s$ are unknown constants and $t\in\mathbb{R}_+$. We proceed to find a_k , b_k and $c_k{}'s$ as follows. Using the initial conditions (3.7) in (3.8), we obtain $a_0=1,\ b_0=0$ and $c_0=0$.

Using the fact that

$${}^cD^q x(t) = (\sqrt[3]{a})^q z(t)$$

and substituting (3.8) in the above equation we get

$$\sum_{k=0}^{\infty}a_{k+1}\frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)}t^{kq} = (a)^{\frac{q}{3}}\sum_{k=0}^{\infty}c_k t^{kq}.$$

Comparing the coefficients of the same power on both sides we obtain

$$a_{k+1} = \frac{(a)^{\frac{q}{3}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)}c_k,\quad \text{for}\ \ k=0,1,2,\ldots.$$

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Similarly by using the equations ${}^cD^q y(t) = (\sqrt[3]{a})^q x(t)$ and ${}^cD^q z(t) = (\sqrt[3]{a})^q y(t)$, we get

$$b_{k+1} = \frac{(a)^{\frac{q}{3}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_k$$

and

$$c_{k+1} = \frac{(a)^{\frac{q}{3}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)} b_k \quad \text{for} \quad k = 0,1,2,\ldots.$$

For $k=0$, we get $a_1 = 0$, $b_1 = \dfrac{(a)^{\frac{q}{3}}}{\Gamma(1+q)}$, $c_1 = 0.$

$k=1$, yields $a_2 = 0$, $b_2 = 0$, $c_2 = \dfrac{(a)^{\frac{2q}{3}}}{\Gamma(1+2q)}.$

For $k=2$, we have $a_3 = \dfrac{a^q}{\Gamma(1+3q)}$, $b_3 = 0$, $c_3 = 0.$

By continuing this process successively, we finally get the solutions as

$$x(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{3kq}}{\Gamma(1+3kq)} = N^q_{3,0}(t,a) \qquad (\text{ say }) \qquad (3.9)$$

$$y(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{(3k+1)q}}{\Gamma(1+(3k+1)q)} = N^q_{3,1}(t,a) \quad (\text{ say }) \quad (3.10)$$

$$z(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{(3k+2)q}}{\Gamma(1+(3k+2)q)} = N^q_{3,2}(t,a) \quad (\text{ say }). \quad (3.11)$$

The proof is complete.

We next provide another method to verify the same result as in the Theorem 3.1.

Verification.

Consider the IVP (3.1) - (3.2). Let the solutions of the IVP (3.1) - (3.2) be given by

$$x(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{3kq}}{\Gamma(1+3kq)}, \quad t \in \mathbb{R}_+$$

$$y(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \quad t \in \mathbb{R}_+$$

and

$$z(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{(3k+2)q}}{\Gamma(1+(3k+2)q)}, \quad t \in \mathbb{R}_+.$$

To verify them, we consider

$${}^cD^q\big(x(t)\big) = {}^cD^q\left[\sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{3kq}}{\Gamma(1+3kq)}\right], \quad t \in \mathbb{R}_+$$

$$= {}^cD^q\left[1 + \frac{a^q t^{3q}}{\Gamma(1+3q)} + \frac{a^{2q} t^{6q}}{\Gamma(1+6q)} + \frac{a^{3q} t^{9q}}{\Gamma(1+9q)} + \cdots\right]$$

$$= \frac{a^q t^{2q}}{\Gamma(1+2q)} + \frac{a^{2q} t^{5q}}{\Gamma(1+5q)} + \frac{a^{3q} t^{8q}}{\Gamma(1+8q)} + \cdots$$

Differentiating both sides using the Caputo derivative, we get

$${}^cD^{2q} x(t) = \frac{a^q t^q}{\Gamma(1+q)} + \frac{a^{2q} t^{4q}}{\Gamma(1+4q)} + \frac{a^{3q} t^{7q}}{\Gamma(1+7q)} + \cdots$$

Again differentiating both sides using the Caputo derivative, we obtain

$${}^cD^{3q} x(t) = a^q + \frac{a^{2q} t^{3q}}{\Gamma(1+3q)} + \frac{a^{3q} t^{6q}}{\Gamma(1+6q)} + \cdots$$

$$= a^q\left[1 + \frac{a^q t^{3q}}{\Gamma(1+3q)} + \frac{a^{2q} t^{6q}}{\Gamma(1+6q)} + \cdots\right]$$

$$= a^q \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{3kq}}{\Gamma(1+3kq)} = a^q x(t)$$

or

$${}^cD^{3q} x(t) - a^q x(t) = 0.$$

Also the initial condition $x(0)=1$ is satisfied.

Hence

$$x(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{3kq}}{\Gamma(1+3kq)}, \quad t \in \mathbb{R}_+$$

is the solution of IVP (3.1) - (3.2).

Similarly we can verify that

$$y(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \quad t \in \mathbb{R}_+$$

and

$$z(t) = \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}\,t)^{(3k+2)q}}{\Gamma(1+(3k+2)q)}, \quad t \in \mathbb{R}_+$$

are the solutions of the IVP (3.1) - (3.2).

This completes the verification.

In this setup, the Wronskian property is as follows:

Theorem 3.2. Let $x(t)$, $y(t)$ and $z(t)$ be three solutions of the CFDE (3.1). These three solutions are linearly independent on $\mathbb{R}_+$ if and only if the Wronskian

$$W(x,y,z)(t) \neq 0, \quad \text{for every} \quad t \in \mathbb{R}_+.$$

Proof. Let the Wronskian of the solutions $x(t), y(t)$ and $z(t)$ of the CFDE (3.1), be such that $W(t) \neq 0$. We show that $x(t)$, $y(t)$ and $z(t)$ are linearly independent solutions. If possible, assume that

$x(t),\ y(t)$ and $z(t)$ are linearly dependent. Then there exists a linear combination of solutions as $ax(t)+by(t)+cz(t) = 0,\ a, b, c$ and $t \in \square_{+}$ where a,b,c are not simultaneously zero. Suppose $a \neq 0$. Then

$$x(t) = -\frac{b}{a}y(t) - \frac{c}{a}z(t).$$

By setting $h = -\frac{b}{a}$ and $l = -\frac{c}{a}$, we get

$$x(t) = hy(t) + lz(t).$$

Now consider the Wronskian

$$W(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ {}^cD^q(x(t)) & {}^cD^q(y(t)) & {}^cD^q(z(t)) \\ {}^cD^{2q}(x(t)) & {}^cD^{2q}(y(t)) & {}^cD^{2q}(z(t)) \end{vmatrix}$$

$$= \begin{vmatrix} hy(t) + lz(t) & y(t) & z(t) \\ h\,{}^cD^q y(t) + l\,{}^cD^q z(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ h\,{}^cD^{2q} y(t) + l\,{}^cD^{2q} z(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix}$$

$$= \begin{vmatrix} hy(t) & y(t) & z(t) \\ h\,{}^cD^q y(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ h\,{}^cD^{2q} y(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix} +$$

$$\begin{vmatrix} lz(t) & y(t) & z(t) \\ l\,{}^cD^q z(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ l\,{}^cD^{2q} z(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix}$$

$$= 0.$$

Hence $W(t) = 0$, which is a contradiction. Therefore the solutions $x(t), y(t)$ and $z(t)$ are linearly independent.

To obtain a sufficient condition assume that $x(t), y(t)$ and $z(t)$ are linearly independent solutions. We show that $W(t) \neq 0$.

If possible $W(t) = 0$ for some $t \in \square_{+}$.

Then

$$\begin{vmatrix} x(t) & y(t) & z(t) \\ {}^cD^q x(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ {}^cD^{2q} x(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix} = 0.$$

Then there exists a linear combination of columns as

$$a\begin{bmatrix} x(t) \\ {}^cD^q x(t) \\ {}^cD^{2q} x(t) \end{bmatrix} + b\begin{bmatrix} y(t) \\ {}^cD^q y(t) \\ {}^cD^{2q} y(t) \end{bmatrix} + \begin{bmatrix} z(t) \\ {}^cD^q z(t) \\ {}^cD^{2q} z(t) \end{bmatrix} = 0$$

where a,b and c are not simultaneously zero.

If $a \neq 0$ then $x(t) = -\frac{b}{a}y(t) - \frac{c}{a}z(t).$

This implies that $x(t), y(t)$ and $z(t)$ are linearly dependent, which is a contradiction as the assumption is that these solutions are linearly independent.

Hence $W(t) \neq 0.$

This completes the proof.

The following theorem gives a relation between the solutions of the CFDE (3.1).

Theorem 3.3. Let $x(t), y(t)$ and $z(t)$ be three linearly independent increasing solutions of the CFDE (3.1) on the interval $[t_0, T]$. Then the Wronskian

$$W(t) \leq W(t_0) + \frac{K(t-t_0)^q}{q}, \quad t \geq t_0, \quad \frac{2}{3} < q \leq 1 \tag{3.12}$$

where

$$k = a^{\frac{4q}{3}}\left(3x^2(T)z(T) + 3y^2(T)x(T) + 3z^2(T)y(T)\right).$$

Proof. Since $x(t), y(t)$ and $z(t)$ are three linearly independent increasing solutions of the CFDE (3.1), we have Wronskian

$$W(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ {}^cD^q x(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ {}^cD^{2q} x(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix}$$

$$= \begin{vmatrix} x(t) & y(t) & z(t) \\ (a)^{\frac{q}{3}} z(t) & (a)^{\frac{q}{3}} x(t) & (a)^{\frac{q}{3}} y(t) \\ (a)^{\frac{2q}{3}} y(t) & (a)^{\frac{2q}{3}} z(t) & (a)^{\frac{2q}{3}} x(t) \end{vmatrix}$$

$$= x(t)\left[a^q x^2(t) - a^q y(t)z(t)\right] -$$

$$y(t)\left[a^q z(t)x(t) - a^q y^2(t)\right] + z(t)\left[a^q z^2(t) - a^q x(t)y(t)\right]$$

$$= a^q\left[x^3(t) + y^3(t) + z^3(t) - 3x(t)y(t)z(t)\right]$$

Operating Caputo fractional differential operator ${}^cD^q$ on both sides, we get

$${}^cD^q W(t) = a^q\, {}^cD^q\left[x^3(t) + y^3(t) + z^3(t) - 3x(t)y(t)z(t)\right]$$

$$= a^q\left[\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}\begin{bmatrix} 3x^2(s)x'(s) \\ +3y^2(s)y'(s) + 3z^2(s)z'(s) \end{bmatrix} ds\right.$$

$$-3\left\{\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x'(s)y(s)z(s)ds + \right.$$

$$\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x(s)y'(s)z(s)ds$$

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$$+\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x(s)y(s)z'(s)ds\Bigg\}\Bigg].$$

Thus

$$^{c}D^{q}W(t)\le a^{q}\Big[3x^{2}(t)\,{}^{c}D^{q}x(t)+3y^{2}(t)\,{}^{c}D^{q}y(t)+3z^{2}(t)\,{}^{c}D^{q}z(t)$$
$$-3\{y(t_0)z(t_0)\,{}^{c}D^{q}x(t)+x(t_0)z(t_0)\,{}^{c}D^{q}y(t)+x(t_0)y(t_0)\,{}^{c}D^{q}z(t)\}\Big]$$
$$=3a^{q}\Bigg[a^{\frac{q}{3}}x^{2}(t)z(t)+y^{2}(t)a^{\frac{q}{3}}x(t)+z^{2}(t)a^{\frac{q}{3}}y(t)$$
$$-\{y(t_0)z(t_0)a^{\frac{q}{3}}z(t)+x(t_0)z(t_0)a^{\frac{q}{3}}x(t)+$$
$$x(t_0)y(t_0)a^{\frac{q}{3}}y(t)\}\Bigg]$$
$$\le 3a^{\frac{4q}{3}}\Big[x^{2}(T)z(T)+y^{2}(T)x(T)+z^{2}(T)y(T)$$
$$-\{y(t_0)z^{2}(t_0)+z(t_0)x^{2}(t_0)+x(t_0)y^{2}(t_0)\}\Big]$$
$$\le a^{\frac{4q}{3}}\Big[3x^{2}(T)z(T)+3y^{2}(T)x(T)+3z^{2}(T)y(T)\Big]=k\ \text{(say)}.$$

The above inequalities follow from the fact that

$$x(t_0)\le x(s)\le x(t)\le x(T),$$
$$y(t_0)\le y(s)\le y(t)\le y(T)$$

and

$$z(t_0)\le z(s)\le z(t)\le z(T)\quad \text{for}\quad t_0\le s\le t\le T.$$

Hence

$$^{c}D^{q}W(t)\le k,$$

where $k=a^{\frac{4q}{3}}\left(3x^{2}(T)z(T)+3y^{2}(T)x(T)+3z^{2}(T)y(T)\right)$.

Now $^{c}D^{q}W(t)\le k$ yields

$$W(t)\le W(t_0)+\frac{k(t-t_0)^{q}}{q},\quad t\ge t_0\quad \text{where}$$

$$k=a^{\frac{4q}{3}}\Big[3x^{2}(T)z(T)+3y^{2}(T)x(T)+3z^{2}(T)y(T)\Big].$$

This completes the proof.

From the above result we deduce the following Corollary.

Corollary 3.4. If $x(t), y(t)$ and $z(t)$ are linearly independent increasing solutions of the CFDE (3.1) on the interval $[\,0\,,\,\mathrm{T}\,]$, then

$$x^{3}(t)+y^{3}(t)+z^{3}(t)-3x(t)y(t)z(t)\le 1+3a^{\frac{q}{3}}\left(x^{2}(T)z(T)+y^{2}(T)x(T)+z^{2}(T)y(T)\right)\frac{t^{q}}{q}.$$

Proof. The result follows by taking $t_0=0$ in the Theorem 3.3.

Now we present the addition formulae for solutions of third order CFDE (3.1).

Addition Formulae. We show that the solution $\left(x(t), y(t), z(t)\right)$ of CFDS (3.6) possesses the properties

$$x(t+\eta)=x(\eta)x(t)+z(\eta)y(t)+y(\eta)z(t), \tag{3.13}$$
$$y(t+\eta)=y(\eta)x(t)+x(\eta)y(t)+z(\eta)z(t), \tag{3.14}$$
$$z(t+\eta)=z(\eta)x(t)+y(\eta)y(t)+x(\eta)z(t),\ t,\eta\in\mathbb{R}_{+}. \tag{3.15}$$

To prove these properties we use the method of linear algebra.

If $\left(x(t), y(t), z(t)\right)$ is a solution of the CFDS (3.6) then $(x(t+\eta), y(t+\eta), z(t+\eta))$, $\eta\in\mathbb{R}_{+}$ also satisfies CFDS (3.6) with different initial conditions. Now these solutions can be expressed in terms of $x(t), y(t)$ and $z(t)$ in the following form

$$x(t+\eta)=p_1x(t)+p_2y(t)+p_3z(t), \tag{3.16}$$
$$y(t+\eta)=q_1x(t)+q_2y(t)+q_3z(t), \tag{3.17}$$
$$z(t+\eta)=r_1x(t)+r_2y(t)+r_3z(t), \tag{3.18}$$

where $p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2$ and r_3 are constants to be chosen appropriately for a given value of $\eta\ge 0$.

Consider

$$x(t+\eta)=p_1x(t)+p_2y(t)+p_3z(t).$$

For

$$t=0\quad \text{we get}\quad p_1=x(\eta).$$

Also

$$^{c}D^{q}x(t+\eta)=p_1\,{}^{c}D^{q}x(t)+p_2\,{}^{c}D^{q}y(t)+p_3\,{}^{c}D^{q}z(t).$$

This implies

$$z(t+\eta)=p_1z(t)+p_2x(t)+p_3y(t).$$

For

$$t=0\quad \text{we get}\quad p_2=z(\eta).$$

Operating Caputo fractional differential operator $^{c}D^{q}$ on both sides we get

$$^{c}D^{q}z(t+\eta)=p_1\,{}^{c}D^{q}z(t)+p_2\,{}^{c}D^{q}x(t)+p_3\,{}^{c}D^{q}y(t).$$

This gives

$$y(t+\eta)=p_1y(t)+p_2z(t)+p_3x(t).$$

For

$$t=0\quad \text{we get}\quad p_3=y(\eta).$$

Here we have used the initial conditions (3.7). Substituting the values of p_1, p_2 and p_3 in (3.16) we get

$$x(t+\eta)=x(\eta)x(t)+z(\eta)y(t)+y(\eta)z(t).$$

Similarly we can show that

$$y(t+\eta)=y(\eta)x(t)+x(\eta)y(t)+z(\eta)z(t),$$

and

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$$z(t+\eta)=z(\eta)x(t)+y(\eta)y(t)+x(\eta)z(t).$$

From these relations, by taking $\eta=t$ we get

$$x(2t)=x^2(t)+2z(t)y(t),$$
$$y(2t)=z^2(t)+2x(t)y(t),$$

and

$$z(2t)=y^2(t)+2z(t)x(t),\ \ \eta,\ \ t\in\mathbb{R}_+.$$

These results may be easily used to obtain the values of $x(3t)$, $y(3t)$ and $z(3t)$ and many similar relations.

Similar to the Euler's formulae for the second order CFDE [7], we can obtain the Euler's formulae for the third order CFDE (3.1).

Euler's Formulae. The solutions of the CFDE (3.1) are $E_q(a^{\frac{q}{3}}t^q)$, $E_q(a^{\frac{q}{3}}\omega t^q)$ and $E_q(a^{\frac{q}{3}}\omega^2 t^q)$ where $a^{\frac{q}{3}}, a^{\frac{q}{3}}\omega, a^{\frac{q}{3}}\omega^2\left(\omega=\dfrac{-1+\sqrt{3}i}{2}\right)$ are the roots of the characteristic equation $\lambda^3-a^q=0\ (a>0)$. We express $E_q(a^{\frac{q}{3}}t^q)$, $E_q(a^{\frac{q}{3}}\omega t^q)$ and $E_q(a^{\frac{q}{3}}\omega^2 t^q)$ in terms of $N^q_{3,0}(t,a), N^q_{3,1}(t,a)$ and $N^q_{3,2}(t,a)$ respectively as follows:

$$(i)E_q(a^{\frac{q}{3}}t^q)=\sum_{k=0}^{\infty}\frac{a^{\frac{kq}{3}}t^{kq}}{\Gamma(1+kq)},\quad t\in\mathbb{R}_+$$

$$=1+\frac{a^{\frac{q}{3}}t^q}{\Gamma(1+q)}+\frac{a^{\frac{2q}{3}}t^{2q}}{\Gamma(1+2q)}+\frac{a^q t^{3q}}{\Gamma(1+3q)}+\frac{a^{\frac{4q}{3}}t^{4q}}{\Gamma(1+4q)}+\frac{a^{\frac{5q}{3}}t^{5q}}{\Gamma(1+5q)}+\frac{a^{2q}t^{6q}}{\Gamma(1+6q)}+\cdots$$

$$=1+\frac{a^q t^{3q}}{\Gamma(1+3q)}+\frac{a^{2q}t^{6q}}{\Gamma(1+6q)}+\cdots+\frac{a^{\frac{q}{3}}t^q}{\Gamma(1+q)}+\frac{a^{\frac{4q}{3}}t^{4q}}{\Gamma(1+4q)}+\cdots+\frac{a^{\frac{2q}{3}}t^{2q}}{\Gamma(1+2q)}+\frac{a^{\frac{5q}{3}}t^{5q}}{\Gamma(1+5q)}+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{(\sqrt[3]{a}t)^{3kq}}{\Gamma(1+3kq)}+\sum_{k=0}^{\infty}\frac{(\sqrt[3]{a}t)^{(3k+1)q}}{\Gamma(1+(3k+1)q)}+\sum_{k=0}^{\infty}\frac{(\sqrt[3]{a}t)^{(3k+2)q}}{\Gamma(1+(3k+2)q)}$$

$$=N^q_{3,0}(t,a)+N^q_{3,1}(t,a)+N^q_{3,2}(t,a).$$

$$(ii)E_q(a^{\frac{q}{3}}\omega t^q)=\sum_{k=0}^{\infty}\frac{a^{\frac{kq}{3}}\omega^k t^{kq}}{\Gamma(1+kq)},\quad t\in\mathbb{R}_+$$

$$=1+\frac{a^{\frac{q}{3}}\omega t^q}{\Gamma(1+q)}+\frac{a^{\frac{2q}{3}}\omega^2 t^{2q}}{\Gamma(1+2q)}+\frac{a^q\omega^3 t^{3q}}{\Gamma(1+3q)}+\frac{a^{\frac{4q}{3}}\omega^4 t^{4q}}{\Gamma(1+4q)}+\frac{a^{\frac{5q}{3}}\omega^5 t^{5q}}{\Gamma(1+5q)}+\frac{a^{2q}\omega^6 t^{6q}}{\Gamma(1+6q)}+\cdots$$

$$=1+\frac{a^q t^{3q}}{\Gamma(1+3q)}+\frac{a^{2q}t^{6q}}{\Gamma(1+6q)}+\cdots+\omega\left(\frac{a^{\frac{q}{3}}t^q}{\Gamma(1+q)}+\frac{a^{\frac{4q}{3}}t^{4q}}{\Gamma(1+4q)}+\cdots\right)+\omega^2\left(\frac{a^{\frac{2q}{3}}t^{2q}}{\Gamma(1+2q)}+\frac{a^{\frac{5q}{3}}t^{5q}}{\Gamma(1+5q)}+\cdots\right)$$

$$=\sum_{k=0}^{\infty}\frac{(\sqrt[3]{a}t)^{3kq}}{\Gamma(1+3kq)}+\omega\sum_{k=0}^{\infty}\frac{(\sqrt[3]{a}t)^{(3k+1)q}}{\Gamma(1+(3k+1)q)}+\omega^2\sum_{k=0}^{\infty}\frac{(\sqrt[3]{a}t)^{(3k+2)q}}{\Gamma(1+(3k+2)q)}$$

$$=N^q_{3,0}(t,a)+\omega N^q_{3,1}(t,a)+\omega^2 N^q_{3,2}(t,a).$$

$$(iii)E_q(a^{\frac{q}{3}}\omega^2 t^q)=\sum_{k=0}^{\infty}\frac{a^{\frac{kq}{3}}\omega^{2k}t^{kq}}{\Gamma(1+kq)},\quad t\in\mathbb{R}_+$$

$$=1+\frac{a^{\frac{q}{3}}\omega^2 t^q}{\Gamma(1+q)}+\frac{a^{\frac{2q}{3}}\omega^4 t^{2q}}{\Gamma(1+2q)}+\frac{a^q\omega^6 t^{3q}}{\Gamma(1+3q)}+\frac{a^{\frac{4q}{3}}\omega^8 t^{4q}}{\Gamma(1+4q)}+\frac{a^{\frac{5q}{3}}\omega^{10}t^{5q}}{\Gamma(1+5q)}+\frac{a^{2q}\omega^{12}t^{6q}}{\Gamma(1+6q)}+\cdots$$

$$=1+\frac{a^q t^{3q}}{\Gamma(1+3q)}+\frac{a^{2q}t^{6q}}{\Gamma(1+6q)}+\cdots+\omega^2\left(\frac{a^{\frac{q}{3}}t^q}{\Gamma(1+q)}+\frac{a^{\frac{4q}{3}}t^{4q}}{\Gamma(1+4q)}+\cdots\right)+\omega\left(\frac{a^{\frac{2q}{3}}t^{2q}}{\Gamma(1+2q)}+\frac{a^{\frac{5q}{3}}t^{5q}}{\Gamma(1+5q)}+\cdots\right)$$

$$= \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}t)^{3kq}}{\Gamma(1+3kq)} + \omega^2 \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}t)^{(3k+1)q}}{\Gamma(1+(3k+1)q)} + \omega \sum_{k=0}^{\infty} \frac{(\sqrt[3]{a}t)^{(3k+2)q}}{\Gamma(1+(3k+2)q)}$$

$$= N_{3,0}^{q}(t,a) + \omega^2 N_{3,1}^{q}(t,a) + \omega N_{3,2}^{q}(t,a).$$

Thus we obtain the following relations

$$E_q\left(a^{\frac{q}{3}} t^q\right) = N_{3,0}^{q}(t,a) + N_{3,1}^{q}(t,a) + N_{3,2}^{q}(t,a) \quad (3.19)$$

$$E_q(a^{\frac{q}{3}} \omega t^q) = N_{3,0}^{q}(t,a) + \omega N_{3,1}^{q}(t,a) + \omega^2 N_{3,2}^{q}(t,a) \quad (3.20)$$

and

$$E_q(a^{\frac{q}{3}} \omega^2 t^q) = N_{3,0}^{q}(t,a) + \omega^2 N_{3,1}^{q}(t,a) + \omega N_{3,2}^{q}(t,a), \quad t \in \mathbb{R}_+. \quad (3.21)$$

The equations (3.19), (3.20) and (3.21) are three Euler's forms for the solutions of the CFDE (3.1).
By solving (3.19), (3.20) and (3.21) we obtain

$$N_{3,0}^{q}(t,a) = \frac{1}{3} E_q(a^{\frac{q}{3}} t^q) + \frac{1}{3} E_q(a^{\frac{q}{3}} \omega t^q) + \frac{1}{3} E_q(a^{\frac{q}{3}} \omega^2 t^q), \quad (3.22)$$

$$N_{3,1}^{q}(t,a) = \frac{1}{3} E_q(a^{\frac{q}{3}} t^q) + \frac{\omega^2}{3} E_q(a^{\frac{q}{3}} \omega t^q) + \frac{\omega}{3} E_q(a^{\frac{q}{3}} \omega^2 t^q), \quad (3.23)$$

$$N_{3,2}^{q}(t,a) = \frac{1}{3} E_q(a^{\frac{q}{3}} t^q) + \frac{\omega}{3} E_q(a^{\frac{q}{3}} \omega t^q) + \frac{\omega^2}{3} E_q(a^{\frac{q}{3}} \omega^2 t^q), \quad t \in R^+. \quad (3.24)$$

Here the three solutions of the CFDE (3.1) are expressed in terms of Mittag - Leffler's forms.

IV. Extended Generalized Fractional Hyperbolic Like Functions Through n^{th} Order CFDE

The results obtained in Section 3 can be generalized to n^{th} order CFDE. In this section we study the solutions of the n^{th} order CFDE.

Consider the $(nq)^{th}$ order, $\frac{n-1}{n} < q \le 1$ fractional IVP of the form

$$^{c}D^{nq} x(t) - a^q x(t) = 0 \quad (4.1)$$

with initial conditions

$$x(0) = 1, \ ^{c}D^{q}x(0) = 0, \ ^{c}D^{2q}x(0), \ldots, ^{c}D^{(n-1)q}x(0) = 0 \quad (4.2)$$

where $(n-1) < nq \le n, \quad n \in N$ fixed.

Theorem 4.1. The general solution of the CFDE (4.1) is given by $c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t)$ where $c_1, c_2, \ldots, c_n$ are arbitrary constants and $x_1(t), x_2(t), \ldots, x_n(t)$ are infinite series solutions of the form

$$\begin{aligned} x_1(t) &= \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{nkq}}{\Gamma(1+nkq)} \\ x_2(t) &= \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{(nk+1)q}}{\Gamma(1+(nk+1)q)} \\ &\vdots \\ x_n(t) &= \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{(nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)}, \quad t \in \mathbb{R}_+. \end{aligned} \quad (4.3)$$

Proof. We transform the IVP (4.1) - (4.2) to a system of equations of q^{th} order, $\frac{n-1}{n} < q \le 1$ by taking $\alpha = nq$ and setting

$$^{c}D^{q} x_1(t) = a^{\frac{q}{n}} x_n(t), \ ^{c}D^{q} x_2(t) = a^{\frac{q}{n}} x_1(t), \ ^{c}D^{q} x_3(t) = a^{\frac{q}{n}} x_2(t), \ldots, \ ^{c}D^{q} x_n(t) = a^{\frac{q}{n}} x_{n-1}(t). \quad (4.4)$$

with initial conditions

$$x_1(0) = 1, \ x_2(0) = 0, \ldots, \ x_n(0) = 0. \quad (4.5)$$

Let

$$x(t) = \sum_{k=0}^{\infty} a_{1k} t^{kq}, \ x_2(t) = \sum_{k=0}^{\infty} a_{2k} t^{kq}, \ldots, \ x_n(t) = \sum_{k=0}^{\infty} a_{nk} t^{kq} \quad (4.6)$$

where $a_{ik}{}'s, i = 1, 2, \ldots, n, k = 0, 1, \ldots, \infty$ are unknown constants and $t \in \mathbb{R}_+$.
From the initial conditions (4.5) we have

$$a_{10} = 1, \ a_{20} = 0, \ldots, \ a_{n0} = 0.$$

Now consider the equation

$$^{c}D^{q} x_1(t) = a^{\frac{q}{n}} x_n(t).$$

Substituting (4.6) in the above equation we get

$$^{c}D^{q}\left[\sum_{k=0}^{\infty} a_{1k} t^{kq}\right] = a^{\frac{q}{n}} \sum_{k=0}^{\infty} a_{nk} t^{kq}$$

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which gives

$$\sum_{k=0}^{\infty} a_{1(k+1)} \frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)} t^{kq} = a^{\frac{q}{n}} \sum_{k=0}^{\infty} a_{nk} t^{kq}.$$

Further comparison of the coefficients of the same power yields

$$a_{1(k+1)} = \frac{a^{\frac{q}{n}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{nk} \quad \text{for} \quad k=0,1,2,\ldots.$$

Similarly using

$${}^{c}D^{q} x_2(t) = a^{\frac{q}{n}} x_1(t),\ {}^{c}D^{q} x_3(t)$$

$$= a^{\frac{q}{n}} x_2(t),\ldots,\ {}^{c}D^{q} x_n(t) = a^{\frac{q}{n}} x_{n-1}(t)$$

we get

$$a_{2(k+1)} = \frac{a^{\frac{q}{n}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{1k},\ a_{3(k+1)}$$

$$= \frac{a^{\frac{q}{n}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{2k},\ldots,a_{n(k+1)}$$

$$= \frac{a^{\frac{q}{n}}\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{(n-1)k} \quad \text{for} \quad k=0,1,2,\ldots.$$

Using the above recursive relations, we obtain the values of $a_{11}, a_{12},\ldots, a_{21}, a_{22},\ldots, a_{n1}, a_{n2},\ldots$ and finally the solutions are given by

$$x_1(t) = \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{nkq}}{\Gamma(1+nkq)} = N_{n,0}^{q}(t,a) \quad (say)$$

$$x_2(t) = \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{(nk+1)q}}{\Gamma(1+(nk+1)q)} = N_{n,1}^{q}(t,a) \quad (say) \qquad (4.7)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_n(t) = \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{(nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)} = N_{n,n-1}^{q}(t,a) \quad (say).$$

The proof is complete.

At this stage, we consider a suitable notation to conveniently represent such infinite series. The notation is as follows.

$$N_{n,r}^{q}(t,a) = \sum_{k=0}^{\infty} \frac{\left(a^{\frac{1}{n}} t\right)^{(nk+r)q}}{\Gamma(1+(nk+r)q)},$$

$$r=0,1,2,\ldots,(n-1),\ n\in N,\ t\in\mathbb{R}_{+}.$$

These solutions are called as extended generalized fractional hyperbolic like functions.

Now we state and prove a theorem which relates the Wronskian and the solutions of the CFDE (4.1).

Theorem 4.2. Let $x_1(t), x_2(t),\ldots, x_n(t)$ be n solutions of the CFDE (4.1). These n solutions are linearly independent on $\mathbb{R}_{+}$ if and only if the Wronskian $W(t)\neq 0$ for every $t\in\mathbb{R}_{+}$.

Proof. Let there be a point t_1 in $\mathbb{R}_{+}$ such that $W(t_1)\neq 0.$ Assume that there are n constants $c_1, c_2,\ldots c_n$ such that $c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t) = 0,\ t\in\mathbb{R}_{+}.$

To show that $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent, we must show that $c_1 = c_2 = \ldots c_n = 0.$ At $t=t_1$ in $\mathbb{R}_{+}$ we have

$$c_1 x_1(t_1) + c_2 x_2(t_1) + \ldots + c_n x_n(t_1) = 0$$

$$c_1\, {}^{c}D^{q} x_1(t_1) + c_2\, {}^{c}D^{q} x_2(t_1) + \ldots + c_n\, {}^{c}D^{q} x_n(t_1) = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_1\, {}^{c}D^{(n-1)q} x_1(t_1) + c_2\, {}^{c}D^{(n-1)q} x_2(t_1) + \ldots + c_n\, {}^{c}D^{(n-1)q} x_n(t_1) = 0$$

These are n simultaneous homogeneous equations in $c_1, c_2,\ldots c_n$ as unknown coefficients. Since the determinant formed by the coefficients of the n equations $W(t_1)\neq 0,$ it is clear that $c_1 = c_2 = \ldots c_n = 0.$ Therefore the solutions are linearly independent.

To obtain a sufficient condition assume that the solutions $x_1(t), x_2(t),\ldots, x_n(t)$ are linearly independent. We show that Wronskian $W(t)\neq 0.$

Suppose if possible that $\mathrm{W}(\mathrm{t}) = 0$ for some $t\in\mathbb{R}_{+}$.

Then

$$\begin{vmatrix} x_1(\mathrm{t}) & x_2(t) & \cdots & x_n(t) \\ {}^{c}D^{q} x_1(t) & {}^{c}D^{q} x_2(t) & \cdots & {}^{c}D^{q} x_n(t) \\ \vdots & \vdots & \vdots & \vdots \\ {}^{c}D^{(n-1)q} x_1(t) & {}^{c}D^{(n-1)q} x_2(t) & \cdots & {}^{c}D^{(n-1)q} x_n(t) \end{vmatrix} = 0.$$

Then there exists a linear combination of columns as

$$c_1 \begin{bmatrix} x_1(t) \\ {}^{c}D^{q} x_1(t) \\ \vdots \\ {}^{c}D^{(n-1)q} x_1(t) \end{bmatrix} + c_2 \begin{bmatrix} x_2(t) \\ {}^{c}D^{q} x_2(t) \\ \vdots \\ {}^{c}D^{(n-1)q} x_2(t) \end{bmatrix} + \cdots$$

$$+\ c_n \begin{bmatrix} x_n(t) \\ {}^{c}D^{q} x_n(t) \\ \vdots \\ {}^{c}D^{(n-1)q} x_n(t) \end{bmatrix} = 0$$

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where $c_1, c_2, \ldots, c_n$ are not simultaneously zero. If $c_1 \neq 0$ then $x_1(t) = -\frac{c_2}{c_1}x_2(t) - \frac{c_3}{c_1}x_3(t) - \cdots - \frac{c_n}{c_1}x_n(t)$. This implies that $x_1(t),\ x_2(t),\ \ldots,\ x_n(t)$ are linearly dependent solutions, which is a contradiction. Thus $W(t) \neq 0.$

This completes the proof.

V. Conclusion

This paper deals with a family of specific type of CFDE involving a parameter '*a*'. The solutions of the *3rd* order and *nth* order CFDEs are obtained analytically. Inequality results between the solutions of the *3rd* order CFDE are obtained. Further properties of these solutions are studied.

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