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Modeling and Stabilization of Cart Triple Link Inverted Pendulum using LQR Controller Incorporating Degree of Stability

Kalyani Sharma[1], Vikas Sahu[2]
[1]M.E. Scholar, Department of Electronics & Telecommunication Engineering, Shri Shankaracharya Technical Campus, Bhilai [C.G.], India
[2]Assistant Professor, Department of Electronics & Telecommunication Engineering, Shri Shankaracharya Technical Campus, Bhilai [C.G.], India

Abstract— *In this paper a mathematical model for the cart triple inverted pendulum system is first presented using Lagrange equation in details. This model is then used to design a controller based on the LQR method to maintain the triple inverted pendulum on a cart around its unstable equilibrium position using single control input. The stability, controllability and observability are investigated and the choice of weights in LQR also discussed. Main focus is to introduce how to build the mathematical model and the analysis of the system's performance. The system is simulated in MATLAB environment and the simulation results establish the satisfactory performance of LQR controller in stabilizing the system.*

Keywords—*Linear quadratic regulator, triple link inverted pendulum, matlab, system performance.*

I. Introduction

The inverted pendulum is a classical bench mark problem in dynamics and control theory. Its dynamics is similar to many real world systems for instance humanoid robots, missile launchers, human walking, Segway, automatic aircraft landing system, biped locomotive machines, flexible space structures and many more industrial applications. Control of the Inverted pendulum system has been one of the challenging topics in control theory [1].It is highly unstable, highly nonlinear, non-minimum phase and under actuated system. Because of these features it reveals many interesting system theoretic properties. Additionally we consider inverted pendulum mounted on a cart. So, the physical constraints on the track position, control voltage etc. adds up to the complexity of the design. It is widely used as benchmark problem for testing control algorithms. A variation on this problem includes multiple links. The complexity increases as the number of link increases.

There are various types of pendulum such as, the simple inverted pendulum, the rotary inverted pendulum, double inverted pendulum, the rotary double inverted pendulum [2].In this paper we have considered a triple link inverted pendulum mounted on a cart. The TLIP system is a SIMO (Single input multi Output) system. This kind of pendulum system is difficult to control due to the inherent instability and nonlinear behavior. In this paper, a continuous time linear quadratic regulator (LQR) with degree of stability is used for stabilizing the triple link inverted pendulum. The main aim of our work is to improve the overall performance of the cart TLIP system. a state space design approach is used as it well suited to the control of multiple outputs as we have. The concept of stability, controllability and observability is also discussed. We will attempt to control both the pendulum's angle and cart's position. The rest of the paper deals with section 2 explain the mathematical modeling of the cart TLIP system. Section 3 is associated with designing of LQR controller with degree of stability, section 4 with simulation and results and section 5 with conclusion.

II. Mathematical Modeling

The mathematical model of the cart TLIP system is obtained by Euler's Lagrange equation. [4], [5] The schematic of the system is shown below. The pendulum consists of three links of different lengths which are mounted on a cart., u is external action; x is displacement of cart; θ_1, θ_2, θ_3 are the angles of the lower, middle, and upper pendulum bars respectively with respect to the vertical line; m_0 is the mass of the cart; m_1, m_2, m_3 are the Centre masses of the lower, middle, upper pendulum bar respectively; L1,L2,L3 are the length of the lower, middle ,upper pendulum respectively.l1,l2,l3 are the centroid of the lower, middle, upper pendulum bar respectively f_0 is the friction factor of cart and track, f1 is the friction factor of lower pendulum and cart,f2 is the friction factor of middle and lower pendulum,f3is the friction factor of middle and upper pendulum; J_1, J_2, J_3 are the rotary inertia of lower, middle and upper pendulum bar respectively, K_s is the overall system's input conversion gain

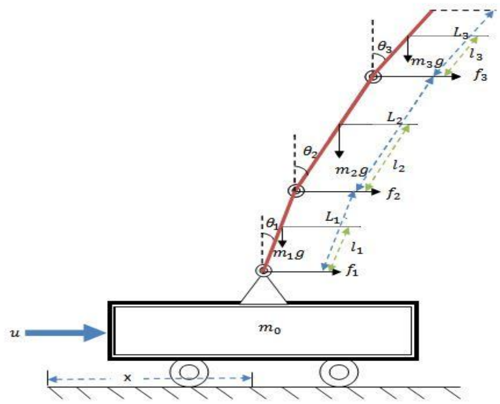

Fig. 1. Schematic of cart TLIP.

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The generalized Euler-Lagrange equation is given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) \quad = -\left(\frac{\partial D}{\partial \dot{q}_i}\right) \qquad (1)$$

Where, Lagrange function (Lagrangian),
L = T – V
T = Total Kinetic energy of the system
V = Total Potential energy of the system
W = Work done against friction (Dissipative forces)
q_1, q_2……………, q_s are the generalized coordinates of the system
The kinetic, potential and friction dissipation energies are given as follows: the kinetic energy is

$$K = \ \frac{1}{2}m_0\dot{x}^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}m_1\left[\left(\dot{x} + l_1\cos\theta_1.\dot{\theta}_1\right)^2 + \left(l_1\sin\theta_1.\dot{\theta}_1\right)^2\right] + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}m_2\left[\left(\dot{x} + L_1\cos\theta_1.\dot{\theta}_1 + l_2\cos\theta_2.\dot{\theta}_2\right)^2 + \left(L_1\sin\theta_1.\dot{\theta}_1 + l_2\sin\theta_2.\dot{\theta}_2\right)^2\right] + \frac{1}{2}J_3\dot{\theta}_3^2 + \frac{1}{2}m_3\left(\dot{x} + L_1\cos\theta_1.\dot{\theta}_1 + L_2\cos\theta_2.\dot{\theta}_2 + l_3\cos\theta_3.\dot{\theta}_3\right)^2 + \frac{1}{2}m_3\left(L_1\sin\theta_1.\dot{\theta}_1 + L_2\sin\theta_2.\dot{\theta}_2 + l_3\sin\theta_3.\dot{\theta}_3\right)^2 \qquad (2)$$

The total potential energy of the system is

$$P = P_0 + P_1 + P_2 + P_3$$
$$P = (m_1gl_1 + m_2gL_1 + m_3gL_1)\cos\theta_1 + (m_2gl_2 + m_3gL_2) \qquad (3)$$

The Lagrange function of Triple Link Inverted Pendulum System is:

$$L\left(x, \theta_1, \theta_2, \theta_3, \dot{x}, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\right) = T - V \qquad (4)$$

The work done against friction

$$W = \frac{1}{2}f_0\dot{x}^2 + \frac{1}{2}f_1\dot{\theta}_1^2 + \frac{1}{2}f_2(\dot{\theta}_2 - \dot{\theta}_1)^2 + \frac{1}{2}f_3(\dot{\theta}_3 - \dot{\theta}_2)^2 \qquad (5)$$

The mathematical model of the triple inverted pendulum is constructed based on the Lagrange equations [12]

$$M(\theta_1, \theta_2, \theta_3)\begin{bmatrix}\ddot{x}\\ \ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{bmatrix} + N(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2\dot{\theta}_3)\begin{bmatrix}\dot{x}_1\\ \dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} = G(u, \theta_1, \theta_2, \theta_3) \qquad (6)$$

The non-linear model described by (6) is linearized about upright position i.e.

$$\theta_1 = \ \theta_2 = \theta_3 = 0$$

and

$$sin(\theta_1) = sin(\theta_2) = sin(\theta_3) \approx 0$$

$$G_0 = [K_s\ 0\ 0\ 0]^T$$

$$M_0 = \begin{bmatrix} k_0 & k_1 & k_2 & m_3l_3 \\ k_1 & k_3 & k_2L_1 & m_3L_1l_3 \\ k_2 & a_2L_1 & b_2 & m_3L_2l_3 \\ m_3l_3 & m_3L_1l_3 & m_3L_2l_3 & J_3 + m_3l_3^2 \end{bmatrix}$$

$$N_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_1g & 0 & 0 \\ 0 & 0 & k_2g & 0 \\ 0 & 0 & 0 & m_3l_3g \end{bmatrix}$$

$$F_0 = \begin{bmatrix} -f_0 & 0 & 0 & 0 \\ 0 & -f_1 - f_2 & f_2 & 0 \\ 0 & f_2 & -f_2 - f_3 & f_3 \\ 0 & 0 & f_3 & -f_3 \end{bmatrix}$$

Where, the coefficients are given as:

$$\left.\begin{aligned} k_0 &= \ m_0 + m_1 + m_2 + m_3 \\ k_1 &= \ m_1l_1 + m_2L_1 + m_3L_1 \\ k_2 &= m_2l_2 + m_3L_2 \\ k_3 &= J_1 + m_1l_1^2 + m_2L_1^2 + m_3L_1^2 \\ k_4 &= J_2 + m_2l_2^2 + m_3L_2^2 \end{aligned}\right\} \qquad (7)$$

The linear model of the triple link inverted pendulum is represented in state-space form as follows:

$$\left.\begin{aligned} \dot{X} \ &= \ AX + BC \\ \dot{X} \quad &= \begin{bmatrix} 0_{4\times4} & I_{4\times4} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} 0_{4\times1} \\ B_2 \end{bmatrix}U \\ Y &= C\ X \end{aligned}\right\} \qquad (8)$$

Where, $A_{21} = M_0^{-1} \times N_0, A_{22} = M_0^{-1} \times F_0, B_2 = M_0^{-1} \times G_0$
The state vector is defined by:

$$X = [x\ \ \theta_1\ \theta_2\ \theta_3\dot{x}\ \ \dot{\theta}_1\ \dot{\theta}_2\ \ \dot{\theta}_3]^T \qquad (9)$$

The coefficient matrices of state equation (8) of cart triple inverted pendulum after putting the parameter values mentioned in APPENDIX are as follows:

$$A_{21} = \begin{bmatrix} 0 & -15.1861 & 2.7420 & -0.3023 \\ 0 & 147.1379 & -82.4455 & 9.0896 \\ 0 & -182.6242 & 194.3229 & -41.2087 \\ 0 & 37.9468 & -77.6656 & 45.8506 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} -5.2356 & 0.0136 & -0.0107 & 0.0027 \\ 17.1258 & -0.1778 & 0.2052 & -0.0822 \\ -6.8497 & 0.2967 & -0.4352 & 0.2246 \\ 1.4233 & -0.0924 & 0.2138 & -0.1355 \end{bmatrix}$$

$$B = [0\quad 0\quad 0\quad 0\quad 3.7397\quad -12.2326\quad 4.8926\quad -1.0166]^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

III. CHARACTERISTICS ANALYSIS

After obtaining the mathematical model of the system features, we need to analyze the stability; controllability and observability of system's in order to further understand the characteristics of the system

A. Stability Analysis

If the closed loop poles are all located in the left half of the s-plane, the system must be stable, otherwise the system is unstable. In MATLAB to strike a linear time invariant system, the characteristics roots can be obtained by eig(A,B)

TABLE 2. Eigen values of open loop system.

S. No.	Eigen Values of Unstable System
1	0
2	16.9833
3	-18.2763
4	7.3830
5	3.8938
6	-9.0828
7	-1.9199
8	-4.9833

The eigen values of the system matrix A for the system are given in table II. The system is unstable as it has positive Eigen values. The model is simulated in MATLAB and the step response of the open loop TIPS

B. Controllability Analysis

Linear time-invariant controllability systems necessary and sufficient condition is:

Rank (C) =n Where C is controllability matrix given by

$$C= [B\ AB\ A^2B\ldots\ldots\ A^{n-1}B]$$

The dimension of the matrix A is n. In MATLAB, the function ctrb(a,b) is used to test the controllability of matrix ,through the calculation we can see that the system is controllable.

C. Observability Analysis

Linear time-invariant observability systems necessary and sufficient condition is:

Rank (O)=n where O is observability matrix given by

$$O= [C\ CA\ CA^2\ldots\ldots CA^{n-1}]^T$$

In MATLAB, the function obsv(a,b) is used to test the observability of matrix ,through the calculation we can see that the system is observable.

IV. Design Of LQR Controller

Our LTI plant model with directly measurable state x is given as

$$\dot{X} = AX + BU \tag{11}$$

For the controlled system described the quadratic performance index is given by:

$$J = \frac{1}{2}\int_0^{\infty}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt \tag{12}$$

Where Q is appositive semi definite matrix, R is positive definite matrix. For our work we have incorporated degree of stability $\boldsymbol{\alpha}$. All closed-loop poles are to the left of $-\boldsymbol{\alpha}$. So we define a new performance index cost function

$$J = \frac{1}{2}\int_0^{\infty}[e^{2\alpha T}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}]dt \tag{13}$$

Where $R = R^T > 0, Q = Q^T \geq 0, M = M^T \geq 0$ are weight matrixes.So defining a new state variable and control

$$u^*(t) = -Kx(t); \tag{14}$$

$$K = -R^{-1}(t)B^T(t)P(t) \tag{15}$$

Where, K is Kalman gain, P(t) is positive definite matrix is the optimal solution of matrix differential Riccati equation (DRE)

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) \tag{16}$$

The Riccati matrix equation or algebraic Riccati equation (ARE) is given as;

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{17}$$

Since we have incorporated degree of stability α, our equation will be modified as follows:

$$P(A + \alpha I) + (A + \alpha I)^TP - PBR^{-1}B^TP + Q = 0 \tag{18}$$

Using the LQR method, the effect of optimal control depends on the selection of weighting matrices Q and R, if Q and R selected not properly, it make the solution cannot meet the actual system performance requirements. In general, Q and Rare taken the diagonal matrix, the current approach for selecting weighting matrices Q and R is simulation of trial, after finding a suitable Q and R, it allows the use of computers to find the optimal gain matrix K easily.

V. Simulation Results

First we have chosen Q matrix as $Q = C * C^T$

Q 1= diag ([1 0 1 0 1 0 1 0]), R=1 and Degree of stability α = 0.1

The optimal feedback gain matrix is

K = [-0.7392 -78.3976 192.3977 -145.2670 -3.8914 -1.0185 7.8722 -20.8247]

The step response of this system is shown in fig 2(a), 2(b), 2(c) and 2(d)

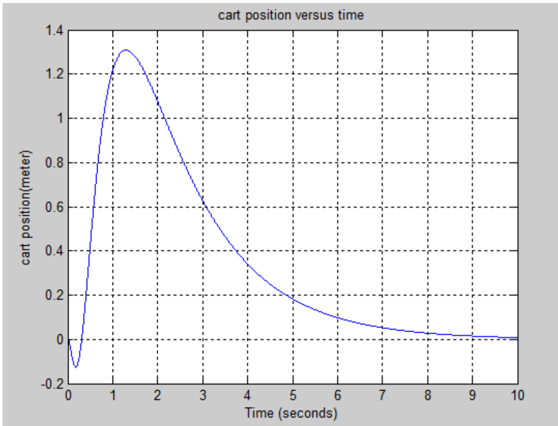

Fig. 2 (a). Step response of cart position.

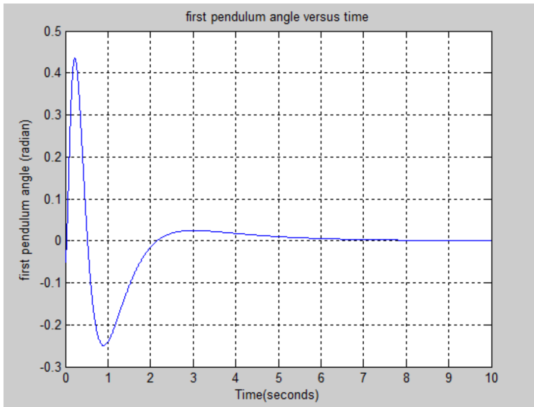

Fig. 2(b). Step response of 1st angle.

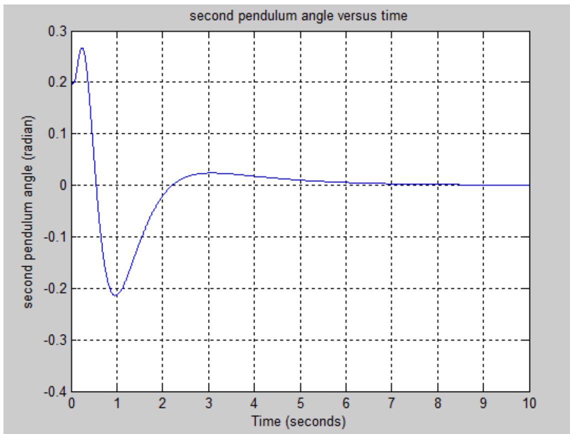

Fig. 2(c). Step response of 2nd angle.

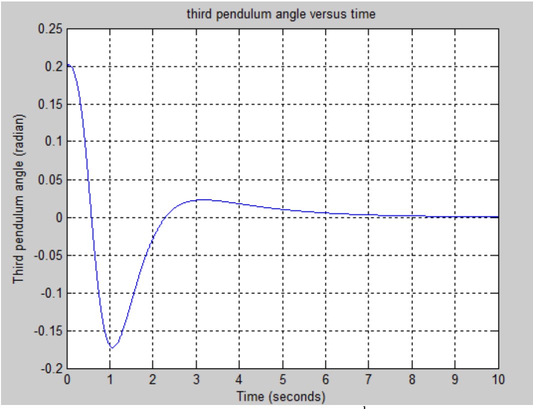

Fig. 2(d). Step response of 3rd angle.

The settling time and rise time are too large. To minimize the rise time and settling time we have changed the value of the diagonal matrix Q. This is done by iteration method. Replace the element of matrix (Q) by

Q2 = diag ([1400 1200 400 200 0 0 0 0]), R = 1, Degree of stability α=0.1

$[K, P, E] = lqr(A, B,Q, R)$ where E is the open loop eigen value

Now, the optimal feedback gain matrix is

K= [-42.1825 -93.3181 355.7447 -566.4401 -51.9743 -17.6816 -9.3716 -95.9650]

TABLE II. Eigen values of closed loop system.

S. No.	Eigen Values of stable system
1	-18.3087+10.9661i
2	-18.3087-10.9661i
3	-13.3256+4.7427i
4	-13.3256-4.7427i
5	-5.3860+1.4948i
6	-5.3860-1.4948i
7	-2.3955+1.2390i
8	-2.3955-1.2390i

From table 2 it is clear that the closed loop system is stable since all poles have negative real parts i.e. all poles lie in left half of the s plane. After using LQR controller i.e. Adding gain K and matrix B the system is stable in nature. The improved step response of the system is shown in fig 3(a), 3(b), 3(c), 3(d).

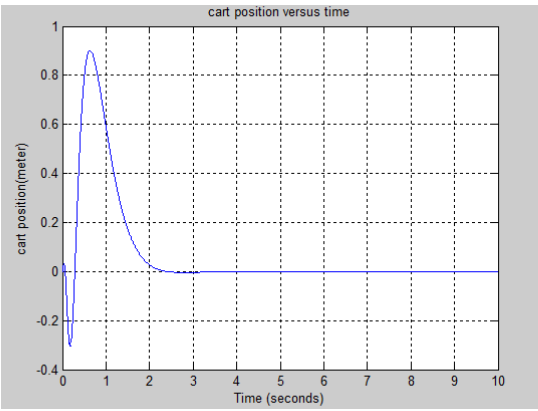

Fig. 3(a). Improved step response of cart position.

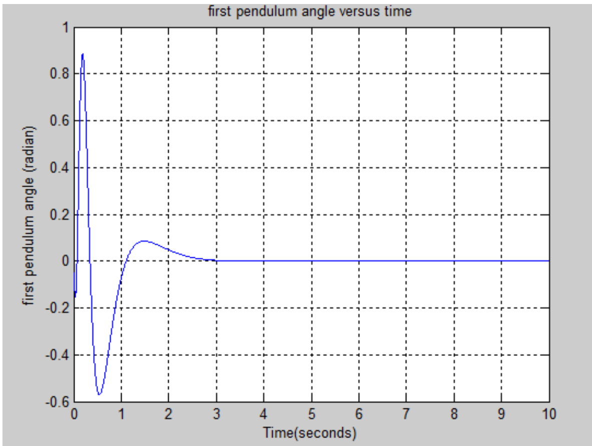

Fig. 3(b): Improved step response of 1st angle.

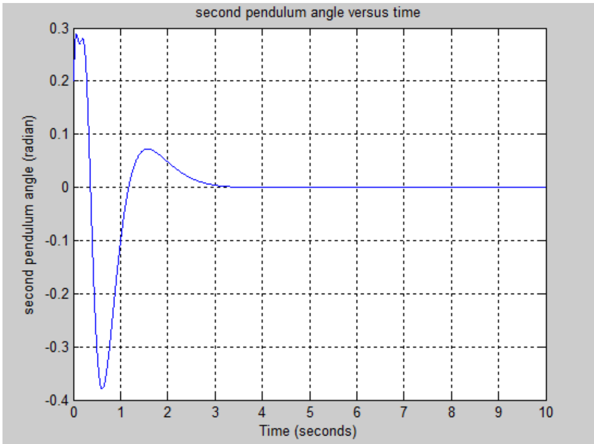

Fig. 3(c): Improved step response of 2nd angle.

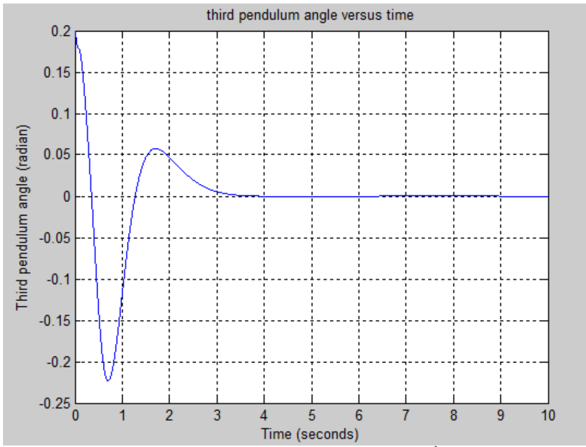

Fig. 3(d). Improved step response of 3rd angle.

TABLE III. Reference tracking performance.

Parameters of TLIP System	Settling time (seconds)		Rise time (seconds)	
	Q1	Q2	Q1	Q2
Cart Position	7.6997	2.0892	0.0036	1.1587e-12
1st pendulum angle	5.1592	2.4597	0.0215	0.0050
2nd pendulum angle	5.9175	2.8133	0.1399	0.0635
3rd pendulum angle	6.3556	3.0724	0.3162	0.2617

VI. CONCLUSION

In this paper a LQR controller is successfully designed for a cart triple link inverted pendulum successfully. From table it is clear that LQR controller for modified value of diagonal matrix Q2 is better than default value of diagonal matrix Q1. The settling time for our controller (Q2) for cart position, first, second and third pendulum angles are 72.86%, 52.32%, 52.45% and 51.65% less than default value of Q1. Similarly improvement is shown in rise time. The rise time for our controller (Q2) for cart position, first, second and third pendulum angles are 0.3%, 76.74%, 54.61% and17.23% less than default value of Q1 The performance of the proposed LQR controller is found to be better and the settling time is also small. Simulation results clearly establish the effectiveness of the proposed controller as the system performance and stability are satisfactory**.** The performance of the proposed LQR controller is found to be good and settling time is also small. In this paper the techniques to reduce the settling time and rise time of the system is also discussed. Simulation results clearly show the effectiveness of the proposed controller

VII. Appendix

m_0= 2.4 Kg

m_1=1.323 Kg	m_2=1.389 Kg	m_3= 0.8655 Kg	
L_1=0.402 m,	L_2=0.332 m	L_3=0.72 m	
l_1=0.2449 m	l_2=0.193 m	l_3=0.3405 m	
J_1=.0119 Kgm^2	J_2=.0069 Kgm^2	J_3=.0291 Kgm^2	
f_0=13.611 Nsm	f_1=.0045 Nsm	f_2=0.0045 Nsm	
f_3=0.0045 Nsm	K_s= 9.722 NV	g=9.81ms^{-2}	[8]

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