International Research Journal of Advanced Engineering and Science

ISSN: 2455-9024

Characterization of Various Spaces Via δsg^* - Closed Sets

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Abstract**—***The aim of this paper is to introduce the class of δsg^*-closed sets and obtain the characterizations of $T_{1/2}$ space, partition space, semi-weakly Hausdorff space, R_1 space and Hausdorff space via δsg^*-closed sets.*

Keywords**—** *δsg^*-closed set, δgs-closed set, gδs-closed set, gs-closed set.*

I. Introduction

In 1963, Levine introduced and studied the concepts of semi-open sets in topological spaces as a weaker form of open sets [12]. The study of generalized closed sets was initiated by Levine in order to extend the topological properties of closed sets to a larger family of sets in 1970 [11]. Dontchev et.al. [4], Sudha et.al. [16] and Meena et.al. [13] introduced various g-closed sets using δ-closed sets. In 2007, J.H.Park et.al. introduced and studied two concepts namely gδs -closed and δgs -closed sets using δ-semi closure and proved that the class of δgs-closed sets is weaker than the class of gδs -closed sets [8, 9]. In this paper, we introduce and study the class of δsg^*-closed sets, which is weaker than the class of δ-semi closed sets and is stronger than the classes of gδs -closed sets and δgs -closed sets. The spaces in which the concepts of g-closed and closed sets coincide are called $T_{1/2}$-spaces [12]. $T_{1/2}$-spaces are precisely the spaces in which singleton are open or closed. T_b and T_d spaces are introduced by Devi et.al., [3] and Semi-$T_{1/2}$ -spaces by Bhattacharya et.al., [2]. We use δsg^* -closed sets to obtain new characterization of semi-weakly Hausdorff spaces which are the spaces with semi-$T_{1/2}$-semi regularization [8].

II. Preliminaries

We list some definitions in a topological space (X, τ) which are useful in the following sections. The interior (δ-interior) and the closure (δ-closure) of a subset A of (X, τ) are denoted by int(A) (δ-int(A)) and cl(A) (δ-cl(A)) respectively. Throughout the present paper (X, τ) represents non-empty topological space on which no separation axiom is defined, unless otherwise mentioned.

Definition 2.1 A subset A of (X, τ) is called a

1) a semi open set [12] if $A \subseteq cl(int(A))$
2) a δ-open set [17] if it is a union of regular open sets
3) a δ-semi open [10] if $A \subseteq cl(\delta\text{-}int(A))$

The complement of a semi open (resp. δ -open, δ-semi open) set is called a semi closed (resp. δ -closed, δ-semi closed) set. The δ-semi interior of a subset A of (X, τ) is the union of all δ-semi open sets contained in A and is denoted by δ-sint(A) and the δ-semi closure of a subset A of (X, τ) is the intersection of all δ-semi closed sets containing A and is denoted by δ-scl(A).

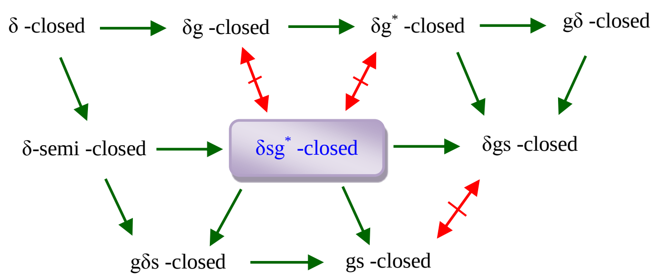
Definition 2.2 A subset A of a space (X, τ) is said to be

(a) generalized closed (briefly g-closed) [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(b) generalized semi-closed (briefly gs-closed) [1] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(c) generalized δ-semi-closed (briefly gδs-closed) [8] if $\delta\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(d) δ-generalized closed (briefly δg-closed) [4] if $\delta\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(e) generalized δ-closed (briefly gδ-closed) [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ).
(f) δ-generalized star -closed (briefly δg^* -closed) [5] if $\delta\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ).
(g) δ-generalized semi-closed (briefly δgs-closed) [9] if $\delta\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ).

III. δsg^* -CLOSED SETS

Definition 3.1 [6] A subset A of a topological space (X, τ) is called δ semi generalized star -closed (briefly δsg^* -closed) set if $\delta\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in X.

Remark 3.2 For a subset of a topological space, from definitions stated above, we have the following diagram of implications:



where none of these implications is reversible as shown by examples in [3,4,9] and the following examples.

Example 3.3 [9] Let X = {a, b, c, d} and τ = {X, ϕ, {a}, {c}, {a,b}, {a,c}, {a,b,c}, {a,c,d}}. Then

a) Set A = {a,c,d}. Then A is δgs -closed but neither gs -closed nor gδs -closed in (X, τ).
b) Set B = {b,c}. Then B is gs -closed but not δgs -closed in (X, τ).
c) Set C = {c}. Then C is δ-semi closed (hence δgs -closed) but not δg^* -closed in (X, τ).

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Example 3.4 Let X = {a, b, c, d} and τ = {X, ϕ, {a}, {c}, {a,b}, {a,c}, {a,b,c}, {a,c,d}}.

1) Set A = {b}. Then A is gδs -closed and gδ -closed but neither δg^* -closed nor δ-semi closed in (X, τ).
2) Set B ={c}. Then B is δ-semi closed (hence δgs) but not gδ -closed in (X, τ).

Example 3.5 Let X = {a, b, c, d} and τ = {X, ϕ, {a}, {b}, {a,b}, {a,b,c}}. Set A = {a,b,c}. Then A is δg^* -closed but not gδs -closed in (X, τ), since A$\in\tau$ but δ-scl(A)=X $\not\subset$ A.

Example 3.6 Let X = {a,b,c}, τ = {X,ϕ,{a}}. Then

1) Set A = {b,c}. Then A is δsg^* -closed but not δ-semi closed.
2) Set B = {a,b}. Then B is gδs -closed but not δsg^* -closed.
3) Set C = {a,c}. Then C is gs -closed but not δsg^* -closed.
4) Let X = {a,b,c}, τ = {X,ϕ,{a},{a,b}}. Then the set {b} is gs -closed but not gδs -closed.
5) Let X = (a,b,c}, τ = {X,ϕ,{a},{b},{a,b}}. Then the set {a} is gδs -closed but not δg^*-closed.

IV. CHARACTERIZATIONS VIA δsg^* -CLOSED SETS

In this section many characterizations are derived via δsg^* -closed sets.

Results 4.1

(1) In semi-regular spaces, the notions of gδs -closed and δgs -closed sets coincide [8, 9].

(2) A space (X, τ) is T_d [2] (resp. T_b [2]) if every gs -closed set is g -closed (resp. Closed).

Definition 4.2 A space with semi -$T_{1/2}$ semi -regularization is called ***semi weakly Hausdorff*** [8].

Now, we observe that in semi-regular $T_{1/2}$-spaces the notions of δsg^* -closed and gs -closed sets coincide.

Theorem 4.3 Let A be a subset of a *$T_{1/2}$ -space* (X, τ) then:

(a) A is δsg^* -closed if and only if A is gδs -closed

(b) If (X, τ) is semi regular then A is δsg^* -closed if and only if A is gs -closed.

(c) If in addition, (X, τ) is T_b (resp. T_d) A is δsg^* -closed if and only if A is closed (resp. g-closed).

Proof: (a) Let (X, τ) be $T_{1/2}$. Then g-open sets coincide with open sets which leads to (a).

(b) In a semi regular space gδs -closed sets coincide with gs -closed sets [Theorem 2.8 of [8]] Then (a) implies (b).

(c) The proof follows from Theorem 2.8 of [8] and from (a).

Definition 4.4 A ***partition space*** is a space where every open set is closed.

Remark 4.5 In a partition space open sets coincide with δ-open sets and the concepts of δ-closure and δ-semi closure coincide for any set.

Theorem 4.6 For a subset A of a *$T_{1/2}$ partition space* (X, τ) the following are equivalent:

(a) A is δsg^* -closed

(b) A is δg -closed

(c) A is δg^* -closed

(d) A is gδs -closed

(e) A is δgs -closed

Proof: (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) is proved in [Theorem 2.6 of [8]]

(a) $\Leftrightarrow$ (b) In a $T_{1/2}$ -space, g-open sets coincide with open sets and hence by Remark 4.5 the proof follows.

The previous observation leads to the problem of finding the spaces (X, τ) in which the gs-closed sets of (X, τ_s) are δsg^* -closed in (X, τ). While we have not been able to completely resolve this problem, we offer partial solutions. For that reason the spaces with semi-$T_{1/2}$ semi-regularization is called *semi-weakly Hausdorff*. Recall that a space is called *almost weakly Hausdorff* [4] if its semi-regularization is $T_{1/2}$. Clearly almost weakly Hausdorff spaces are semi-weakly Hausdorff, but not conversely.

Example 4.7 [8] Let X = {a, b, c, d} with τ = {X, ϕ, {a}, {b}, {a,b}}. Then (X, τ) is clearly semi-weakly Hausdorff but not almost weakly Hausdorff.

Theorem 4.8 For a subset A of a *semi weakly Hausdorff space* (X, τ) the following are equivalent:

(a) A is gs -closed in (X, τ_s)

(b) A is δ-semi closed in (X, τ)

(c) A is δsg^* -closed in (X, τ).

Proof: (a) $\Rightarrow$ (b) Let A $\subseteq$ X be a gs -closed subset of (X, τ_s). Let x$\in\delta$-scl(A). If {x} is δ-semi open, then x$\in$A. If not then X \ {x} is δ-semi open, since X is semi weakly Hausdorff. Assume that x $\notin$ A. Since A is gs-closed in (X, τ_s), then δ-scl(A) $\subseteq$ X \ {x}, that is x $\notin$ δ-scl(A). By contradiction x $\in$ A. Thus δ-scl(A) = A or equivalently A is δ-semi closed.

(b) $\Rightarrow$ (c) The proof follows from the definition of δsg^* -closed sets.

(c) $\Rightarrow$ (a) Let A $\subseteq$ U, where U is open in (X, τ_s). Then U is δ-open in (X, τ). Every δ-open set is g-open. Since A is δsg^* -closed in (X, τ), δ-scl(A) $\subseteq$ U. Hence by Lemma 7.3 of [14], scl(A) $\subseteq$ U in (X, τ_s). Thus A is gs -closed in (X, τ_s).

Theorem 4.9 For a space (X, τ) the following are equivalent:

(a) Every g -open set of X is a δ-semi closed set

(b) Every subset of X is a δsg^* -closed set.

Proof: (a) $\Rightarrow$ (b) Let A $\subseteq$ U, where U is g-open and A is an arbitrary subset of X. By (a), U is δ-semi closed and thus δ-scl(U) $\subseteq$ U. Thus δ-scl(A) $\subseteq$ δ-scl(U) $\subseteq$ U. Hence A is δsg^* -closed.

(b) $\Rightarrow$ (a) Let U be a g-open set of (X, τ), then by (b) δ-scl(U) $\subseteq$ U or equivalently U is δ-semi closed.

Remark 4.10

(a) Every finite union of δsg^* -closed sets may fail to be a δsg^*-closed set.

(b) Every finite intersection of δsg^* -closed sets may fail to be a δsg^*-closed set.

The following examples support the above remark.

Example 4.11 Let X = {a,b,c}, τ = {X, ϕ, {a},{b},{a,b}}. Consider A = {a} and B = {b} then A and B are δsg^* -closed sets but A∪B = {a,b} is not a δsg^*-closed set in (X, τ).

Example 4.12 Let X = {a, b, c, d}, τ = {X, ϕ, {c}, {a,b}, {a,b,c}}. Consider A = {a,b} and B = {a,d} then A and B are δsg^* -closed sets but A∩B = {a} is not a δsg^* -closed set in (X, τ).

Definition 4.13 A topological space (X, τ) is called an ***R_1 -space*** if every two different points with distinct closures have disjoint neighborhoods.

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Theorem 4.14 For a compact subset A of an R_1 *-topological space* (X, τ) the following conditions are equivalent, when (X, τ) is also $T_{1/2}$.
(a) A is a δsg* -closed set
(b) A is a gs -closed set
Proof:
(a) ⇒ (b) is clear.
(b) ⇒ (a) Let A ⊆ U, where U is g-open in (X, τ). In a $T_{1/2}$ -space, g -open sets coincide with open sets. In R_1-spaces the concepts of closure and δ-closure coincide for compact sets [Theorem 3.6 in [7]]. Thus the rest of the proof follows from the definition of δsg* -closed sets.
Corollary 4.15 In Hausdorff spaces, a finite set is gs -closed if and only if it is δsg* -closed.
Theorem 4.16 Let A be a subset of (X, τ). Then we have if A is δsg* -closed in X, then δ-scl(A) \ A does not contain any non empty closed set converse not true.
Proof: Let F be a closed set such that F ⊆ δ-scl(A) \ A. Then A ⊆ X \ F. Since A is a δsg*-closed set and X \ F is open and hence g-open, δ-scl(A) ⊆ X \ F, that is F ⊆ X \ δ-scl(A). Hence F⊆δ-scl(A) ∩ (X \ δ-scl(A)) = ϕ. This shows that F = ϕ.
For the converse part, the counter example is:
Let X = {a,b,c} and τ = {X, ϕ, {a}}. Let A = {b} then δ-scl(A) \ A = X \ A = {a,c}. Therefore δ-scl(A) \ A does not contain non-empty closed set {b,c}. But A is not δsg* -closed.
Theorem 4.17 If A is δsg* -closed in (X, τ) if and only if δ-scl(A) \ A does not contain any non empty g-closed set in (X, τ).
Proof: Necessity - Suppose that A is δsg* -closed, let F be any g-closed such that F ⊆ δ-scl(A) \ A. Then A ⊆ X \ F and X \ F is g-open in (X, τ). Since A is δsg*-closed set in (X, τ), δ-scl(A) ⊆ X \ F. Thus, F ⊆ X \ δ-scl(A). Therefore, F ⊆ (δ-scl(A) \ A) ∩ (X \ δ-scl(A)) = ϕ. Hence F = ϕ.
Sufficiency - Suppose that A ⊆ U and U is any g-open set in (X, τ). If A is not a δsg*-closed set, then δ-scl(A) ⊆ U and hence δ-scl(A) ∩ (X \ U) ≠ ϕ. We have a non empty g-closed set δ-scl(A) ∩ (X \ U) such that δ-scl(A) ∩ (X \ U) ⊆ δ-scl(A) ∩ (X \ A) = δ-scl(A) \ A which contradicts the hypothesis.
Corollary 4.18 If A is a δsg*-closed subset of (X, τ), then A is δ-semi closed if and only if δ-scl(A) \ A is g-closed.
Theorem 4.19 Let A be a subset of (X, τ). Then we have if A is δsg* -closed in X and A ⊆ B ⊆ δ-scl(A), then B is also a δsg* -closed set.
Proof: Let U be a g-open set of X such that B ⊆ U then A ⊆ U. Since A is a δsg* -closed set, δ-scl(A) ⊆ U. Also since B ⊆ δ-scl(A), δ-scl(B) ⊆ δ-scl(δ-scl(A)) = δ-scl(A). Hence δ-scl(B) ⊆ U. Therefore B is also a δsg*-closed set.
Theorem 4.20 If A is g-open and δsg* -closed in (X, τ), then A is a δ-semi closed set of X.
Proof: If A is g-open and δsg* -closed. Let A ⊆ A, where A is g-open and δ-scl(A) ⊆ A which implies δ-scl(A) = A. Hence A is δ-semi closed.

Theorem 4.21 Let A ⊆ Y ⊆ X. Then
(a) If Y is open in (X, τ) and A is δsg* -closed in X, then A is δsg* -closed relative to Y.
(b) If Y is δsg* -closed and g-open in (X, τ) and A is δsg* -closed relative to Y, then A is δsg* -closed in X.
Proof: (a) Let A ⊆ Y∩G, where G is g-open. Since A is δsg*-closed in (X, τ), δ-scl(A) ⊆ Y∩G ⊆ G, which implies Y∩δ-scl(A) ⊆ Y∩G which is g-open. Therefore Y∩δ-scl(A) ⊆ G. Then A is δsg* -closed relative to Y, as Y∩δ-scl(A) is the δ-scl(A) relative to Y. That is Y∩δ-scl(A) = $\delta\text{-scl}_Y(A)$.
(b) Let G be a g-open subset of (X, τ) such that A ⊆ G. Then A ⊆ G∩Y. Since A is δsg* -closed relative to Y, then δ-scl(A) ⊆ G∩Y, i.e. δ-scl(A)∩Y ⊆ G∩Y from Theorem 4.2.25 of [15] and Theorem 4.20, δ-scl(A) = δ-scl(A∩Y) = δ-scl(A) ∩ δ-scl(Y) = δ-scl(A) ∩ Y. Therefore δ-scl(A) ⊆ G∩Y ⊆ G. Hence A is δsg* -closed in X.

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