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On Quasilinearization Method for Hybrid Caputo Fractional Differential Equations

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***Abstract*— *In this paper we present the methods of Quasilinearization and Generalized Quasilinearization for hybrid Caputo fractional differential equations which are Caputo fractional differential equations with fixed moments of impulse. In order to prove this results we use the weakened assumption of -continuity in place of local Hölder continuity*.

Keywords and Phrases — *Hybrid Caputo fractional differential equations, Quasilinearization, Generalized Quasilinearization, Existence*.

AMS Subject Classification. 34K07, 34A08.

I. Introduction

At present, there has been a significant amount of work done in the theory of fractional differential equations and a number of researchers are concetrating in this promising area due to its significant potential in applications related to Fluid Flow, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Optics and Signal Processing etc. The works of Podlubny [1], Kilbas et al. [2], Lakshmikantham et al. [3] and the references [4-9] project the continued interest in this area.

A lot of scope for the development of the theory of hybrid systems or impulsive differential systems has come due to the invaluable contribution of Lakshmikantham et al. [10]. This is due to the fact that many evolution processes are characterized by the fact that they experience a change of state in a very short duration of time. This abrupt change can be considered as short term perturbations whose duration is negligible. Thus we assume that these perturbations act instantaneously in the form of impulses. Hence hybrid systems form a better model to represent physical phenomena.

Combining these two areas of interest, we consider hybrid fractional differential equations and propose to study existence of solutions. Further the method of Quasilinearization [11] is a flexible mechanism that gives a sequence of iterations that converge quadratically to a solution. In [12] Quasilinearization for IVP of fractional differential equations has been studied and in [13] Generalized Quasilinearization has been developed for IVP of fractional differential equations with local holder continuity. Following the observations in [14] that the results in fractional differential equations can be studied with the weakened hypothesis of C_p or C^q continuity.

Thus in this paper, we give an exposition of the Quasilinearization and Generalized Quasilinearization method for existence and uniqueness of the solutions of an IVP of Hybrid Caputo Fractional Differential Equation with the weakened hypothesis [16], [17].

II. Preliminaries

In this section, the basic definitions and results concenring the existence and stability for Hybrid Caputo fractional differential equations with fixed movemnets of Impulse are presented.

We begin with the definition of the class $C_p[[t_0,T],\mathbb{R}]$.

Definition 2.1 *m is said to be C_p continuous if $m\in C_p[[t_0,T],\mathbb{R}]$ that is $m\in C[(t_0,T],\mathbb{R}]$ and $(t-t_0)^p m(t)\in C[[t_0,T],\mathbb{R}]$ with $p+q=1$.*

Definition 2.2 *For $m\in C_p[[t_0,T],\mathbb{R}]$, the Riemann-Liouville derivative of $m(t)$ is defined as*

$$D^q m(t)=\frac{1}{\Gamma(p)}\frac{d}{dt}\int_{t_0}^{t}(t-s)^{p-1}m(s)ds. \tag{2.1}$$

We next state a lemma that is vital for our main result.

Lemma 2.3 *Let $m\in C_p[[t_0,T],\mathbb{R}]$. Suppose that for any $t_1\in[t_0,T]$, we have $m(t_1)=0$ and $m(t)<0$ for $t_0\le t<t_1$, then it follows that*

$$D^q m(t_1)\ge 0. \tag{2.2}$$

We next state the fundamental fractional differential inequality result in the set up of Riemann-Liouville fractional derivative, with a weaker hypothesis from [14].

Theorem 2.4 *Let*
$v,w\in C_p[[t_0,T],\mathbb{R}],\ f\in C[[t_0,T]\times\mathbb{R},\mathbb{R}]$ *and*

$(i)\ D^q v(t)\le f(t,v(t))$

and

$(ii) D^q w(t)\ge f(t,w(t)),$

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$t_0 < t \le T$, with one of the inequalities (i) or (ii) being strict. Then $v^0 < w^0$, where $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$ implies that

$$v(t) < w(t),\ t_0 \le t \le T. \tag{2.3}$$

The next result deals with the inequality theorem for non strict inequalities.

Theorem 2.5 *Let* $v, w \in C_p[[t_0,T],\mathbb{R}],\ f \in C[[t_0,T]\times\mathbb{R},\mathbb{R}]$ *and*

$(i)\ D^q v(t) \le f(t, v(t))$

and

$(ii) D^q w(t) \ge f(t, w(t)),$

$t_0 < t \le T$. Assume f satisfies the Lipschitz condition

$$f(t,x) - f(t,y) \le L(x-y), \quad x \ge y,\ L > 0. \tag{2.4}$$

Then, $v^0 < w^0$, where $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$, implies

$v(t) \le w(t),\ t \in [t_0, T]$.

We now define a C^q -continuous function.

Definition 2.6 *u is said to be C^q continuous that is $u \in C^q[[t_0,T],\mathbb{R}]$ iff the Caputo derivative of u denoted by ${}^cD^q u$ exists and satisfies*

$${}^cD^q u(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}u'(s)ds. \tag{2.5}$$

We note that the Caputo and Riemann-Liouville derivatives are related as follows:

$${}^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{2.6}$$

We choose to work with the Caputo fractional derivative, since the initial conditions for fractional differential equations are of the same form as those of ordinary differential equations. Further, the Caputo fractional derivative of a constant is zero, which is useful in our work. Consider the IVP for the Caputo fractional differential equation given by

$${}^cD^q x = f(t,x), \quad x(t_0) = x_0, \tag{2.7}$$

for $0 < q < 1,\ f \in C^q[[t_0,T]\times\mathbb{R}^n,\mathbb{R}^n]$.

If $x \in C^q[[t_0,T],\mathbb{R}^n]$ satisfies (2.7), then it also satisfies the Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))ds, \tag{2.8}$$

for $t_0 \le t \le T$.

We now state the comparison theorem for the Caputo fractional differential equation using the same weaker hypothesis.

Theorem 2.7 *Assume that $m \in C^q[[t_0,T],\mathbb{R}]$ and*

${}^cD^q m(t) \le g(t, m(t)),\quad t_0 \le t \le T,$

where $g \in C[[t_0,T]\times\mathbb{R},\mathbb{R}]$. Let $r(t)$ be the maximal solution of the IVP

$${}^cD^q u = g(t,u),\ u(t_0) = u_0, \tag{2.9}$$

existing on $[t_0,T]$ such that $m(t_0) \le u_0$. Then we have $m(t) \le r(t),\ t_0 \le t \le T$.

III. Impulsive Fractional Differential Equations

In this section, we begin with the basic definitions given in [15], where in the existence and stability results for hybrid Caputo fractional differential equation with fixed moments of impulse are studied.

Definition 3.1 *Let $0 \le t_0 < t_1 < t_2 < ... < t_k < ...$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC_p[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h:(t_{k-1},t_k]\times\mathbb{R}^n \to \mathbb{R}^n$ is C_p - continuous on $(t_{k-1},t_k]\times\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$*

$$\lim_{(t,y)\to(t_k^+,x)} h(t,y) = h(t_k^+, x)$$

exists for $k = 1,2,...,n-1$.

Definition 3.2 *Let $0 \le t_0 < t_1 < t_2 < ... < t_k < ...$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC^q[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h:(t_{k-1},t_k]\times\mathbb{R}^n \to \mathbb{R}^n$ is C^q - continuous on $(t_{k-1},t_k]\times\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$*

$$\lim_{(t,y)\to(t_k^+,x)} h(t,y) = h(t_k^+, x)$$

exists for $k = 1,2,...,n-1$.

Consider the hybrid Caputo fractional differential system defined by

$$\begin{cases} {}^cD^q x = f(t,x), t \ne t_k, \\ x(t_k^+) = I_k(x(t_k)), k = 1,2,3,...,n-1, \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where

$f \in PC[I\times\mathbb{R}^n,\mathbb{R}^n], I_k : \mathbb{R}^n \to \mathbb{R}^n, t \in I = [t_0,T],$
$k = 1,2,...,n-1.$

Definition 3.3 *By a solution of the system (3.1), we mean a PC^q continuous function $x \in PC^q[[t_0,T],R^n]$ such that*

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$$x(t)=\begin{cases} x_0(t,t_0,x_0), t_0 \le t \le t_1, \\ x_1(t,t_1,x_1^+), t_1 < t \le t_2, \\ \quad . \\ \quad . \\ \quad . \\ x_k(t,t_k,x_k^+), t_k < t \le t_{k+1}, \\ \quad . \\ \quad . \\ x_{n-1}(t,t_{n-1},x_{n-1}^+), t_{n-1} < t \le T, \end{cases} \tag{3.2}$$

where $0 \le t_0 < t_1 < t_2 < ... < t_{n-1} \le T$ and $x_k(t,t_k,x_k^+)$ is the solution of the IVP of the fractional differential equation

$$\begin{cases} {}^cD^q x = f(t,x), \\ x_k^+ = x(t_k^+) = I_k(x(t_k)). \end{cases}$$

Now we state the basic differential inequality result in this set up from [15], and follow it up with a lemma that guarantees the existance of a unique solution.

Theorem 3.4 *Let u,w* $\in PC^q[[t_0,T],\mathbb{R}]$ *with*

$$\begin{cases} {}^cD^q v(t) \le f(t,v(t)), t \ne t_k, \\ v(t_k^+) \le I_k(v(t_k)), k=1,2,3,...,n-1, \\ v(t_0) \le x_0, \end{cases}$$

and

$$\begin{cases} {}^cD^q w(t) \ge f(t,w(t)), t \ne t_k, \\ w(t_k^+) \ge I_k(w(t_k)), k=1,2,3,...,n-1, \\ w(t_0) \ge x_0, \end{cases}$$

where $f \in PC[I \times \mathbb{R}^n, \mathbb{R}^n]$ and f satisfies the hypothesis

$f(t,x) - f(t,y) \le L(x-y), x \ge y, L > 0$

and I_k is a monotonically nondecreasing function of x. Then $v_0 < w_0$ implies that $v(t) \le w(t), t \in [t_0,T]$.

Lemma 3.5 *The linear non-homogeneous hybrid Caputo fractional differential equation*

$$\begin{cases} {}^cD^q x = -M(x-y) + f(t,y), t \ne t_k, \\ x(t_k^+) = I_k(x(t_k)), k=1,2,3,...,n-1, \\ x(t_0) = x_0, \end{cases}$$

has a unique solution on the interval $[t_0,T]$.

We begin with the definition of lower and upper solutions for the hybrid Caputo fractional differential equation given by

$$\begin{cases} {}^cD^q x = f(t,x), t \ne t_k, \\ x(t_k^+) = I_k(x(t_k)), k=1,2,...,n-1, \\ x(t_0) = x_0, \end{cases} \tag{3.3}$$

where

$f \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n], I_k; \mathbb{R}^n \to \mathbb{R}^n, k=1,2,3,...,n-1$ and $t \in [t_0,T]$.

Definition 3.6 $\alpha, \beta \in PC^q[[t_0,T],\mathbb{R}^n]$ are said to be lower and upper solutions of equation (3.3), if and only if they satisfy the following inequalities

$$\begin{cases} {}^cD^q \alpha \le f(t,\alpha), t \ne t_k, \\ \alpha(t_k^+) \le I_k(\alpha(t_k)), k=1,2,3,...,n-1, \\ \alpha(t_0) \le x_0, \end{cases} \tag{3.4}$$

and

$$\begin{cases} {}^cD^q \beta \ge f(t,\beta), t \ne t_k, \\ \beta(t_k^+) \ge I_k(\beta(t_k)), k=1,2,3,...,n-1, \\ \beta(t_0) \ge x_0, \end{cases} \tag{3.5}$$

respectively.

Lemma 3.7 *Suppose that*

(i) $v_0(t)$ *is the lower solution of the hybrid Caputo fractional differential equation* (3.3).

(ii)Let $v_1(t)$ *be the unique solution of the linear non-homogeneous hybrid Caputo fractional differential equation*

$$\begin{cases} {}^cD^q v_1 = f(t,v_0) + f_x(t,v_0)(v_1 - v_0), t \ne t_k, \\ v_1(t_k^+) = I_k(v_1(t_k)), k=1,2,3,...,n-1, \\ v_1(t_0) = x_0. \end{cases} \tag{3.6}$$

(iii) I_k *is a nondecreasing function in* x, *for each k=1,2,3,...,n-1.*

(iv) f_x *is continuous and Lipschitz on* $[t_0,T]$.

Then $v_0(t) \le v_1(t), \quad t \in [t_0,T]$.

We next state the following lemma without proof.

Lemma 3.8 *Suppose that in Lemma* 3.7, *the assumption (i) and (ii) are replaced by (i)* $w_0(t)$ *be the upper solution of the hybrid Caputo fractional differential equation* (3.3) *and (ii)* $w_1(t)$ *be the unique solution of the linear non-homogeneous hybrid Caputo fractional differential equation*

V. Radhika, "On quasilinearization method for hybrid caputo fractional differential equations," *International Research Journal of Advanced Engineering and Science*, Volume 2, Issue 2, pp. 272-278, 2017.

$$\begin{cases} {}^cD^q w_1 = f(t,w_0) + f_x(t,w_0)(w_1 - w_0), t \neq t_k, \\ w_1(t_k^+) = I_k(w_1(t_k)), k = 1,2,3,...,n-1, \\ w_1(t_0) = x_0, \end{cases}$$

and the assumptions (iii) and (iv) of Lemma 3.7 hold.

Then $w_1(t) \leq w_0(t), t \in [t_0, T]$.

IV. GENERALIZED QUASILINEARIZATION

In this section, we consider hybrid Caputo fractional differential equation where the function on right hand side of the hybrid caputo fractional differential equation is a sum of two functions, one satisfying convex like condition and the other satisfying concave like condition and present the results studied in [17].

We consider the hybrid Caputo fractional differential equation given by

$$\begin{cases} {}^cD^q x = f(t,x) + g(t,x), t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), k = 1,2,3...,n-1, \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

where $f, g \in PC^q[[t_0,T],\mathbb{R}], I_k; \mathbb{R} \to \mathbb{R}$ for each $k = 1,2,3,...,n-1$.

We begin with the definition of natural lower and upper solutions for (4.1).

Definition 4.1 $\alpha, \beta \in PC^q[[t_0,T],\mathbb{R}]$ *are said to be lower and upper solutions of equation* (4.1), *if and only if they satisfy the following inequalities,*

$$\begin{cases} {}^cD^q x \leq f(t,\alpha) + g(t,\alpha), t \neq t_k, \\ \alpha(t_k^+) \leq I_k(\alpha(t_k)), k = 1,2,...,n-1, \\ \alpha(t_0) \leq x_0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} {}^cD^q \beta \geq f(t,\beta) + g(t,\beta), t \neq t_k, \\ \beta(t_k^+) \geq I_k(\beta(t_k)), k = 1,2,3...,n-1, \\ \beta(t_0) \geq x_0, \end{cases} \tag{4.3}$$

respectively.

We first state a couple of Lemmmas that are necessary in the proof of our theorem.

Lemma 4.2 *The linear non-homogeneous hybrid Caputo fractional differential equation*

$$\begin{cases} {}^cD^q x = M(x-y) + f(t,y) + g(t,y), t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), k = 1,2,3,...,n-1, \\ x(t_0) = x_0, \end{cases}$$

has a unique solution on the interval $[t_0, T]$.

Lemma 4.3 *Suppose that*

(i) $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of the hybrid Caputo fractional differential equation (4.1).

(ii) Let $\alpha_1(t)$ and $\beta_1(t)$ be the unique solutions of the linear non-homogeneous hybrid Caputo fractional differential equations

$$\begin{cases} {}^cD^q \alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1 - \alpha_0) + \\ g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1 - \alpha_0), t \neq t_k, \\ \alpha_1(t_k^+) = I_k(\alpha_0(t_k)), k = 1,2,3,...,n-1, \\ \alpha_1(t_0) = x_0, \end{cases} \tag{4.4}$$

and

$$\begin{cases} {}^cD^q \beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1 - \beta_0) + \\ g(t,\beta_0) + g_x(t,\beta_0)(\beta_1 - \beta_0), t \neq t_k, \\ \beta_1(t_k^+) = I_k(\beta_0(t_k)), k = 1,2,3,...,n-1, \\ \beta_1(t_0) = x_0. \end{cases} \tag{4.5}$$

(iii) I_k is nondecreasing function in x for each k=1,2,3,...,n-1.

(iv) f_x, g_x are continuous and Lipschitz on $[t_0, T]$.

Then $\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t)$ on $[t_0, T]$.

We now state the main result of our paper and give a summary of the proof.

Theorem 4.4 *Assume that*

(i) $f, g \in PC[[t_0,T] \times \mathbb{R}, \mathbb{R}]$ and $\alpha_0, \beta_0 \in PC^q[[t_0,T],\mathbb{R}]$ be the natural lower and upper solutions of the IVP for the hybrid Caputo fractional differential equation (4.1) such that $\alpha_0(t) \leq \beta_0(t), t \in [t_0, T]$.

(ii) $f_x(t,x)$ exists, $f_x(t,x)$ is increasing in x for each t, $f(t,x) \geq f(t,y) + f_x(t,y)(x-y), x \geq y$ and $|f_x(t,x) - f_x(t,y)| \leq L_1 |x-y|$,

and further suppose that $g_x(t,x)$ exists, $g_x(t,x)$ is decreasing in x for each t, $g(t,x) \geq g(t,y) + g_x(t,y)(x-y), x \geq y$ and $|g_x(t,x) - g_x(t,y)| \leq L_2 |x-y|$.

(iii) I_k is increasing and lipschitz in x, for each $k = 1,2,3...n-1$.

Then there exist monotone sequences $\{\alpha_n\},\{\beta_n\}$ *such that* $\alpha_n \to \rho, \beta_n \to r \; n \to \infty$ *uniformly and* *monotonically to the unique solution* $\rho = r = x$ *of IVP* (4.1) *on* $[t_0, T]$ *and the convergence is quadratic.*

Proof. Consider the linear hybrid Caputo fractional differential equation given by,

$$\begin{cases} {}^cD^q\alpha_{k+1} = f(t,\alpha_k) + f_x(t,\alpha_k)(\alpha_{k+1}-\alpha_k) \\ +g(t,\alpha_k) + g_x(t,\beta_k)(\alpha_{k+1}-\alpha_k), t \neq t_k, \\ \alpha_{k+1}(t_k^+) = I_k(\alpha_k(t_k)), k = 1,2,3,...,n-1, \\ \qquad \alpha_{k+1}(t_0) = x_0. \end{cases} \tag{4.6}$$

and

$$\begin{cases} {}^cD^q\beta_{k+1} = f(t,\beta_k) + f_x(t,\alpha_k)(\beta_{k+1}-\beta_k) \\ +g(t,\beta_k) + g_x(t,\beta_k)(\beta_{k+1}-\beta_k), t \neq t_k, \\ \beta_{k+1}(t_k^+) = I_k(\beta_k(t_k)), k = 1,2,3,...,n-1, \\ \qquad \beta_{k+1}(t_0) = x_0. \end{cases} \tag{4.7}$$

Then it follows from Lemma 4.2 that the linear hybrid fractional differential equations (4.6) and (4.7) have unique solutions α_{k+1} and β_{k+1} respectively, whenever α_k and β_k are known lower and upper solutions of the IVP (4.1).

Further by setting $k = 0$ in the above system, we apply Lemma 4.3 to obtain that $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ on $[t_0, T]$

Observe that α_0 and β_0 are lower and upper solutions of (4.1) and hence the Lemma can be applied.

We now claim that

$$\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k ... \leq \beta_1 \leq \beta_0 \; on [t_0, T]. \tag{4.8}$$

on $[t_0, T]$.

Since the result is already proved for $n = 0$, we assume that the result holds for $n = k$ and prove it for $n = k+1$, this means that

$$\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1}, \tag{4.9}$$

and that α_k and β_k are solutions of the IVP's

$$\begin{cases} {}^cD^q\alpha_k = f(t,\alpha_{k-1}) + f_x(t,\alpha_k)(\alpha_k-\alpha_{k-1}) + \\ g(t,\alpha_{k-1}) + g_x(t,\beta_k)(\alpha_k-\alpha_{k-1}), t \neq t_k, \\ \alpha_k(t_k^+) = I_k(\alpha_{k-1}(t_k)), k = 1,2,3,...,n-1, \\ \qquad \alpha_k(t_0) = x_0. \end{cases} \tag{4.10}$$

and

$$\begin{cases} {}^cD^q\beta_k = f(t,\beta_{k-1}) + f_x(t,\alpha_k)(\beta_k-\beta_{k-1}) \\ +g(t,\beta_{k-1}) + g_x(t,\beta_k)(\beta_k-\beta_{k-1}), t \neq t_k, \\ \beta_k(t_k^+) = I_k(\beta_{k-1}(t_k)), k = 1,2,3,...,n-1, \\ \qquad \beta_k(t_0) = x_0. \end{cases} \tag{4.11}$$

Now, using the relations (4.8), (4.9) and the hypothesis (ii), we get,

$$\begin{aligned} {}^cD^q\alpha_k &= f(t,\alpha_{k-1}) + f_x(t,\alpha_k)(\alpha_{k-1}-\alpha_k) + g(t,\alpha_{k-1}) \\ &+ g_x(t,\alpha_k)(\alpha_{k-1}-\alpha_k) \\ &\leq f(t,\alpha_k) + g(t,\alpha_k) \end{aligned}$$

and

$$\alpha_k(t_k^+) \leq I_k(\alpha_{k-1}(t_k)) \leq I_k(\alpha_k(t_k))$$

since I_k is an increasing function for each k.

This yields that α_k is a lower solution of (4.1) and further by Lemma 4.2, we obtain that α_{k+1} is a unique solution of (4.10) on $[t_0, T]$ and hence an application of the Lemma 4.3 yields that $\alpha_k \leq \alpha_{k+1}$ on $[t_0, T]$

Similarly, it can be shown that β_k is an upper solution of (4.1) and by Lemma 4.2, that β_{k+1} is a unique solution of (4.11) on $[t_0, T]$ and hence an application of the Lemma 4.3 gives that $\beta_{k+1} \leq \beta_k$ on $[t_0, T]$.

Further working in the lines of the Lemma 4.3, we obtain that $\alpha_{k+1} \leq \beta_{k+1}$ on $[t_0, T]$

Hence by the principle of mathematical induction, we deduce the relation (4.8) and our claim holds. Clearly the sequences are piecewise uniformly bounded by relation (4.8), this also yields that the sequences $\{{}^cD^q\alpha_n\}$ and $\{{}^cD^q\beta_n\}$ are also piecewise uniformly bounded. By Lemma 2.3.2 in [3] we obtain that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous in each subinterval $(t_k, t_{k+1}]$ and therefore by using Ascoli-Arzela Theorem, we conclude that there is a subsequence that converges uniformly on each subinterval $(t_k, t_{k+1}]$. Now since I_k is a continuous function (as lipschitz implies continuity) for each k, this convergence also holds at end points. Thus we obtain a sequence of piecewise C^q-continuous functions $\{\alpha_n(t)\}$ that converge uniformly to $\rho(t)$ on each subinterval $(t_k, t_{k+1}]$ and further $I_k(\alpha_k(t_k^+))$ converges uniformly to $I_k(\rho(t_k^+))$. Similarly, the sequence of iterations $\{\beta_n(t)\}$ converge uniformly to $r(t)$ in each

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subinterval $(t_k, t_{k+1}]$ and further $I_k(\beta_k(t_k^+))$ converges uniformly to $I_k(r(t_k^+))$.

Consider the corresponding hybrid Volterra fractional integrals, we can show that ρ and r are solutions of the IVP(4.1)

Since f_x, g_x exists and is bounded on $[t_0, T]$, we obtain that f and g are Lipschitz and hence the solution is unique.

Thus $\rho = x = r$ on $[t_0, T]$.

To prove that the convergence is quadratic, we set

$$p_{n+1} = x - \alpha_{n+1}$$

Thus for $t \neq t_k$,

$$^{c}D^q p_{n+1} = {^{c}D^q} x - {^{c}D^q}\alpha_{n+1}$$

$= f(t,x) + g(t,x) - f(t,\alpha_n) - g(t,\alpha_n)$

$-[f_x(t,\alpha_n) - g_x(t,\beta_n)]$

$(\alpha_{n+1} - \alpha_n)$

$\leq f_x(t,\xi)p_n + g_x(t,\eta)p_n + [f_x(t,\alpha_n)$

$+g_x(t,\beta_n)](p_{n+1} - p_n)$

where $\alpha_n \leq \xi \leq x$ and $x \leq \eta \leq \beta_n$

Now using the increasing nature of f_x and the decreasing nature of g_x, we get that for $t \neq t_k$,

$$^{c}D^q p_{n+1} \leq L|p_n|_0^2 + Mp_{n+1}$$

where $M = Max(M_1, M_2)$ with $|f_x(t,\alpha_n)| \leq M_1$, $|g_x(t,\beta_n)| \leq M_2$ and $L = Max|L_1, L_2|$

for $t = t_k$,

$$p_{n+1}(t_k^+) \leq K\ p_{n+1}(t_k)$$

Therefore $p_{n+1}(t_k^+) \leq Kp_{n+1}(t_k), t = t_k$.

Since $p_{n+1}(0) = 0$, we arrive at the hybrid Caputo fractional differential equation. Thus we have the hybrid Caputo fractional differential equation

$^{c}D^q p_{n+1} = L|p_n|_0^2 + Mp_{n+1}, t \neq t_k,$

$p_{n+1}(t_k^+) = K_k p_{n+1}(t_k), k = 1,2,3,...n-1,$

$p_{n+1}(0) = 0.$

Now using the solution of the linear non homogeneous fractional differential equation on each subinterval, we get $t \in (t_k, t_{k+1}]$.

For $t \in (t_k, t_{k+1}]$, we have

$$p_{n+1}(t) = K_k.....K_3K_2K_1 \frac{L|p_n|_0^2}{\Gamma(q+1)}(t_1 - t_0)^q$$

$$E_{q,q}(M(t_1 - t_0)^q)E_q(M(t_2 - t_1)^q)$$

$$E_q(M(t_3 - t_2)^q).....E_q(M(t_{k+1} - t_k)^q) + K_k.....K_3K_2$$

$$\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_2 - t_1)^q\ E_{q,q}(M(t_2 - t_1)^q)\ E_q(M(t_3 - t_2)^q).....E_q(M(t_{k+1} - t_k)^q)$$

$$+K_k.....K_3\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_3 - t_2)^q\ E_{q,q}(M(t_3 - t_2)^q).....E_q(M(t_{k+1} - t_k)^q) + ...$$

$$+K_k\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_k - t_{k-1})^q E_{q,q}(M(t_k - t_{k-1})^q)\ E_q(M(t_{k+1} - t_k)^q)$$

$$+\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_{k+1} - t_k)^q\ E_{q,q}(M(t_{k+1} - t_k)^q)$$

$$\leq \frac{L\tilde{K}}{\Gamma(q+1)}E_{q,q}(Ml^q)\Omega|p_n|_0^2$$

where

$\tilde{K} = K_1...K_N$

and

$$\Omega = \sum_{j=1}^{N} l^q[E_q(Ml^q)]^{k-j}$$

Thus

$$|p_{n+1}(t)| \leq \frac{L\tilde{K}}{\Gamma(q+1)}\Omega\, E_{q,q}(Ml^q)|p_n|_0^2$$

This implies the quadratic convergence of the sequence $[\alpha_n(t)]$.

Similarly, we can prove the quadratic convergence of the sequence $\{\beta_n(t)\}$ to the solution $x(t)$ of IVP (4.1).

By taking $g \equiv 0$ in (4.1), we get the hybrid Caputo fractional differential equation (3.3) and the main result reduces to the following theorem which gives us the method of Quasilinearization as in [16].

Theorem 4.5 *Suppose that*

(i) α_0, β_0 be lower and upper solutions of equation (3.3) such that $\alpha_0 \leq \beta_0$ on $[t_0, T]$.

(ii) $f \in PC[[t_0, T] \times \mathbb{R}, \mathbb{R}]$

and

$$f(t,x) \geq f(t,y) + f_x(t,y)(x - y)\ for\ \alpha_0 \leq y \leq x \leq \beta_0;$$

(iii) I_k is continuous and nondecreasing in x,

$k = 1,2,3...n-1.$

(iv) f_x is continuous and Lipschitz on $[t_0, T]$.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *such that* $\alpha_n \to \rho, \beta_n \to r\ n \to \infty$ *uniformly and monotonically to the unique solution* $\rho = r = x$ *of IVP* (3.3) *on* $[t_0, T]$ *and the convergence is quadratic.*

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Remark: It can be observed that if we set $I_k \equiv 0$ for all k, then IVP(4.1) reduces to Caputo fractional differential equation and generalized quasilinearization for there equations has been studied in [13]. Thus these results hold with the weakened hypothesis of C^q-continuity.

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