International Research Journal of Advanced Engineering and Science
ISSN (Online): 2455-9024

Properties of (σ, δ)-Rings

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Abstract— *For a ring R, an automorphism σ of R and δ a σ-derivation of R, we introduce (σ, δ)-ring and (σ, δ)-rigid ring which are the generalizations of σ(∗)-rings and δ-rings, and investigate their properties. Moreover, we prove that a (σ, δ)-ring is 2-primal and for a (σ, δ)-ring R, the prime radical is completely semi-prime.*

Keywords— *Noetherian ring, automorphisms, σ(∗)-rings, δ-rings,(σ, δ)-rings, (σ, δ)-rigid rings and 2-primal.*

I. Introduction and Preliminaries

A ring R always means an associative ring with identity 1 ≠ 0, unless otherwise stated. The prime radical and the set of nilpotent elements of R are denoted by P (R) and N (R) respectively. The field of real numbers is denoted by R, the field of integers by Z, the field of rational numbers by Q and the field of complex numbers by C, unless otherwise stated.

Let R be a ring. This article concerns endomorphisms and derivations of a ring and we also discuss certain types of rings involving endomorphisms and derivations. We begin with the following:

Definition 1.1. (Krempa [10]) An endomorphism σ of a ring R is said to be rigid if aσ(a) = 0 implies that a = 0, for all a ∈ R. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R.

Example 1.2. Let R = C and σ : C → C be defined by σ(a + ib) = a − ib, for all a, b ∈ R. Then σ is a rigid endomorphism of R.

We recall a ring R is σ-rigid if there exists a rigid endomorphism σ of R and σ-rigid rings are reduced rings by Hong et. al. [6]. Properties of σ-rigid rings have been studied in Krempa [10], Hong [6] and Hirano [5].

Definition 1.3. (Kwak [12]) Let R be a ring and σ an endomorphism of R. Then R is said to be σ(∗)-ring if aσ(a) ∈ P (R) implies that a ∈ P (R) for a ∈ R.

Example 1.4. (Example 1 of [12]) Let R = $\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$.

Then P(R) = $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let σ: R → R be defined by

σ($\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$) = $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a σ(∗)-ring. We note that the above ring is not σ-rigid. Let 0 /= a ∈ F. Then

$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$σ($\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$)=$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$≠$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Definition 1.5. [13] An ideal I of a ring R is said to be completely semi-prime if a^2 ∈ I implies that a ∈ I, for a ∈ R.

Definition 1.6. A ring R is said to be 2-primal if and only if P (R) = N (R).

Example 1.7. [4] Let R = F [x] be the polynomial ring over the field F. Then R is 2-primal with P (R) = {0}.

2-primal rings have been studied in recent years and are being treated by authors for different structures. We know that a ring R is 2-primal if the prime radical is completely semi-prime. Note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further detail on 2-primal rings refer to [2, 3, 7, 8, 9, 13, 15]. Furthermore, the concept of completely semi-prime ideals is also studied in this area. Kwak in [12] establishes a relation between a 2-primal ring and a σ(∗)-ring.

Definition 1.8. [14] Let R be a ring, σ an endomorphism of R and δ: R → R an additive map such that

δ(ab) = δ(a)σ(b) + aδ(b), for all a, b ∈ R.

Then δ is a σ-derivation of R.

Example 1.9. Let R = Z[√2]. Then σ : R → R defined as

σ(a + b√2) = a − b√2 for a + b√2 ∈ R

is an endomorphism of R. For any s ∈ R.

Define δ_s : R → R by

δ_s(a + b√2) = (a + b√2)s - s σ(a + b√2) for a + b√2 ∈ R.

Then δ_s is a σ-derivation of R.

Definition 1.10. (Bhat [1]) Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a δ-ring if aδ(a) ∈ P (R) implies that a ∈ P (R).

Note that a δ-ring is without identity, 1 ≠0 as 1.δ(1) = 0, but 1 ≠ 0.

Example 1.11. Let S be a ring without identity and R = S × S with P (R) = {0} (for example we take S = 2Z).

Then σ : R → R is an endomorphism defined by

σ((a, b)) = (b, a).

For any s ∈ R. Define δ_s : R → R by

δ_s((a, b)) = (a, b)s − sσ((a, b)) for (a, b) ∈ R.

Let (a, b)δ_s((a, b)) ∈ P (R).

Then (a, b){(a, b)s − sσ((a, b))} ∈ P (R)

or (a, b){(a, b)s − s(b, a)} ∈ P (R)

i.e. (a, b)(as − bs, bs − sa) ∈ P (R).

Therefore (a(as − bs), b(bs − sa)) ∈ P (R) = {0} which implies that a = 0, b = 0 i.e. (a, b) = (0, 0) ∈ P (R). Thus R is a δ-ring.

In this note we generalize the σ(∗)-rings and δ-rings as follows:

Definition 1.12. Let R be a ring. Let σ be an endomorphism of R and δ a σ- derivation of R. Then R is said to be a (σ, δ)-ring if a(σ(a) + δ(a)) ∈ P (R) implies that a ∈ P (R), for a ∈ R.

Example 1.13. Let R=$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then P(R) = $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

Let σ : R → R be defined by

Meeru Abrol, "Properties of (σ, δ)-Rings," *International Research Journal of Advanced Engineering and Science*, Volume 2, Issue 2, pp. 28-30, 2017.

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R.
For any s ∈ R. Define $\delta_s : R \to R$ by

$$\delta_s(A) = As - s\sigma(A) \text{ for } A \in R.$$

Then δs is a σ-derivation on R.

Now let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$.

A[σ(A) + δ(A)] ∈ P (R) implies that

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\{\sigma(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) + As - s\sigma(A)\} \in P(R)$, which gives on simplifying a = 0, c = 0. Therefore A ∈ P (R).
Hence P (R) is a (σ, δ)-ring.

Remark 1.14.
(1) If δ(a) = 0, then (σ, δ)-ring is a σ(∗)-ring.
(2) If σ(a) = 0, then (σ, δ)-ring is a δ-ring.
(3) If σ(a) = a, δ(a) = 0, then (σ, δ)-ring is completely semi-prime.

Definition 1.15. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ, δ)-rigid ring if a(σ(a) + δ(a)) = 0 implies that a = 0, for a ∈ R.

Example 1.16. Let R = C and σ : C → C be defined by

$$\sigma(a + ib) = a - ib, \text{ for all } a, b \in R.$$

Then σ is an endomorphism on R. Define δ a σ-derivation on R as δ(A) = A − σ(A)
i.e., δ(a + ib) = a + ib − σ(a + ib) = a + ib − (a − ib) = 2ib
Now A[σ(A) + δ(A)] = 0 implies that
(a + ib)[σ(a + ib) + δ(a + ib)] = 0
i.e., (a + ib)[(a − ib) − 2ib] = 0 or (a + ib)(a + ib) = 0 which implies that a = 0, b = 0. Therefore A = a + ib = 0. Hence R is a (σ, δ)-rigid ring.

With this we prove the following:

Theorem A: Let R be a Noetherian integral domain which is also an algebra over Q. Let σ be an automorphism on R and δ a σ-derivation of R. If R is a (σ, δ)- ring, then R is 2-primal. (This has been proved in Theorem (2.2))

Theorem B: Let R be a Noetherian integral domain which is also an algebra over Q. Let σ be an automorphism on R and δ a σ-derivation of R. If R is a (σ, δ)- ring, then P (R) is completely semi-prime. (This has been proved in Theorem (2.5))

Example of a ring satisfying the hypothesis of Theorem A and Theorem B is R = Z. It is a Noetherian integral domain which is also an algebra over Q. Let σ: R → R be defined by σ(a) = 2a. Then it can be seen that σ is an endomorphism of R. For any s ∈ R. Define δ_s: R → R by $\delta_s(a) = as - s\sigma(a)$, for a ∈ R. Then δ_s is a σ-derivation on R. Also R is a (σ, δ)-ring.

II. Proof of Main Results

For the proof of the main results, we need the following:

Proposition 2.1. Let R be an integral domain. Let σ an automorphism of R and δ a σ-derivation of R. Then for u ≠ 0, σ(u) + δ(u) ≠ 0.

Proof. Let 0 ≠ u ∈ R, we show that σ(u)+ δ(u) ≠ 0. Let for 0 ≠ u, σ(u)+ δ(u) = 0 which implies that

(2.1) $$\delta(u) = -\sigma(u)$$

We know that for a, b ∈ R, δ(ab) = δ(a)σ(b) + aδ(b)

By using (2.1), this implies that
δ(ab) = −σ(a)σ(b) + a(−σ(b)) or −σ(ab) = −[a + σ(a)]σ(b). Since σ is an endomorphism of R, this gives
−σ(a)σ(b) = −[a + σ(a)]σ(b) i.e. σ(a) = a + σ(a).
Therefore a = 0, which is not possible. Hence the result.

The above Proposition does not hold if R is not an integral domain.

We now state and prove the main results of this paper in the form of the following Theorems:

Theorem 2.3. Let R be a Noetherian integral domain which is also an algebra over Q. Let σ be an automorphism of R and δ a σ-derivation of R. If R is a (σ, δ)-ring, then R is 2-primal.

Proof. R is a (σ, δ)-ring. We know that a reduced ring is 2-primal. We use the principle of Mathematical Induction to prove that R is a reduced ring. Let for x ∈ R, $x^n = 0$. We use induction on n and show that x = 0. Result is trivially true for n = 1, as $x^n = x^1 = a(\sigma(a) + \delta(a)) = 0$. Now Proposition (2.1), implies that a = 0, Hence x = 0. Therefore the result is true for n = 1. Let us assume that the result is true f+ 1or n = k, i.e. $x^k = 0$ implies that x = 0. Let n = k + 1. Then $x^{k+1} = 0$ which implies that

$$a^{k+1}(\sigma(a) + \delta(a))^{k+1} = 0.$$

Again by Proposition (2.1), we get a = 0. Hence x = 0. Therefore the result is true for n = k + 1 also. Thus the result is true for all n, by the principle of Mathematical Induction. Hence the Theorem.

The converse of the above is not true.

Example 2.4. Let R = F(x), the field of rational polynomials in one variable, x. Then R is 2-primal with P (R) = {0}. Let σ : R → R be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

For r ∈ R, $\delta_r : R \to R$ be a σ-derivation defined as

$$\delta_r(a) = ar - r\sigma(a).$$

Then R is not a (σ, δ)-ring. For take f (x) = xa + b, r = −b/xa.

Towards the proof of the next Theorem, we require the following:

J. Krempa [10] has investigated the relation between minimal prime ideals and completely prime ideals of a ring R. With this he proved the following:

Theorem 2.5. For a ring R the following conditions are equivalent:

(1) R is reduced.
(2) R is semiprime and all minimal prime ideals of R are completely prime.
(3) R is a subdirect product of domains.

Theorem 2.6. Let R be a Noetherian integral domain which is also an algebra over Q. Let σ be an automorphism of R and δ a σ-derivation of R. If R is a (σ, δ)-ring, then P (R) is completely semi-prime.

Proof. As proved in Theorem (2.3), R is a reduced ring and by using Theorem (2.5), the result follows.

The converse of the above is not true.

Example 2.7. Let F be a field, R = F × F. Let $\sigma : R \to R$ be an automorphism defined as $\sigma((a, b)) = (b, a)$, a, b ∈ F. Here P (R) is a completely semi-prime ring, as R is a reduced ring. For r ∈ F. Define $\delta_r : R \to R$ by

$$\delta_r ((a, b)) = (a, b)r - r\sigma((a, b)) \text{ for } a, b \in F.$$

Then δ_r is a σ-derivation on R. Take A = (1, −1), r = 1/2.
Now $A[\sigma(A) + \delta_r (A)]$
$= (1, -1) \{\sigma ((1, -1)) + (1, -1) 1/2 - 1/2 \sigma ((1, -1))\}$
$= (0, 0) \in P (R) = \{0\}$.
But $(1, -1) \neq 0$.
Hence it is not a (σ, δ)-ring.

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