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Bounds for the Moduli of Zeros of Polynomials

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Abstract— *In this paper we find bounds for the moduli of the zeros of a polynomial in terms of its coefficients. The results so obtained generalize many results known already in the field.*

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I. Introduction

A classical result which gives a bound for the moduli of all the zeros of a polynomial in terms of its coefficients is the following known as Cauchy's Theorem [2,4]:

Theorem A. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z|<1+M$, where $M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

Another elegant classical result giving a bound for the moduli of all the zeros of a polynomial with real coefficients is the following known as the Enestrom-Kakeya Theorem [2, 4]:

Theorem B. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then all the zeros of P(z) lie in $|z|\le 1$.

Rgarding the number of zeros of the polynomial in Theorem B in a smaller disc than the closed unit disc, Q.G.Mohammad [3] proved the following result:

Theorem C. The number of zeros of the polynomial P(z) of Theorem B in $|z|\le \frac{1}{2}$ does not exceed $\frac{1}{\log 2}\log\frac{a_n}{a_0}$.

Various extensions, generalizations and refinements of the above results and also, in fact, results of such types are available in the literature. In 2004 Shah and Liman [5] proved the following result:

Theorem D. If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a complex polynomial of degree n such that

$$\sum_{j=0}^{n-1}|a_j| \le |a_n|,$$

then P(z) has all its zeros in $|z|\le 1$.

II. Main Results

The aim of this paper is to weaken the hypothesis in Theorem D and prove

Theorem 1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying

$$\sum_{j=0}^{n}|a_j - a_{j-1}| \le |a_n|, a_{-1}=0.$$

Then all the zeros of P(z) lie in

$$\frac{|a_0|}{2|a_n|-|a_0|} \le |z \le 1|.$$

Remark 1. If we consider the polynomial $P(z)=6z^2+5z+1$, then Theorems B and C are not applicable. By Theorem A, all the zeros of P(z) lie in $|z|<1.83$, whereas by Theorem 1, all the zeros of P(z) lie in $|z|\le 1$. In fact, the zeros of P(z) are $-\frac{1}{2}$ & $-\frac{1}{3}$ whos moduli are less than 1, which is less than the Cauchy's bound.

We prove a more general result than Theorem 1as follows:

Theorem 2. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying

$$\sum_{j=0}^{n}|a_j - a_{j-1}| \le |a_n|, a_{-1}=0.$$

Then all the zeros of P(z) lie in

$$\frac{|a_0|}{|a_n|R^n(R+1)-|a_0|} \le |z| \le 1$$

for $R\ge 1$
and in

$$\frac{|a_0|}{|a_n|R(R^n+1)-|a_0|} \le |z| \le 1$$

for $R\le 1$.

Remark 2. For R=1, Theorem 2 reduces to Theorem 1.

Theorem 3. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying

$$\sum_{j=0}^{n}\left|a_j - a_{j-1}\right| \le \left|a_n\right|, a_{-1} = 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{R}{k}, k > 1, R \ge 1$ does not exceed

$$\frac{1}{\log k}\log\frac{|a_n|R^n(R+1)}{|a_0|}$$

and the number of zeros of P(z) in $|z| \le \frac{R}{k}, k > 1, 0 < R \le 1$ does not exceed

$$\frac{1}{\log k}\log\frac{|a_n|R(R^{n-1}+1)+(1-R)|a_0|}{|a_0|}.$$

Taking R=1 in Theorem 3, we get the following result:

Corollary 1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying

$$\sum_{j=0}^{n}\left|a_j - a_{j-1}\right| \le \left|a_n\right|, a_{-1} = 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{k}, k > 1$ does not exceed

$$\frac{1}{\log k}\log\frac{2|a_n|}{|a_0|}.$$

Taking k=2 in Cor.1, we get the following result:

Corollary 2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying

$$\sum_{j=0}^{n}\left|a_j - a_{j-1}\right| \le \left|a_n\right|, a_{-1} = 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{2|a_n|}{|a_0|} = 1 + \log\frac{|a_n|}{|a_0|}.$$

If also the coefficients $a_j, j = 0,1,......,n$ are all real and positive, we get the following result from Cor.2:

Corollary 3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real positive coefficients satisfying

$$\sum_{j=0}^{n}\left|a_j - a_{j-1}\right| \le \left|a_n\right|, a_{-1} = 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1 + \log\frac{a_n}{a_0}.$$

Remark 3. If $a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$, then $\sum_{j=0}^{n}\left|a_j - a_{j-1}\right| = |a_n| = a_n, |a_0| = a_0$ and we get Theorem C from Cor.3.

III. Lemmas

For the proof of Theorem 3, we need the following results:

Lemma 1: Let f(z) (not identically zero) be analytic for $|z| \le R, f(0) \ne 0$ and $f(a_k) = 0,\ k = 1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(\mathrm{Re}^{i\theta}\right|d\theta - \log|f(0)| = \sum_{j=1}^{n}\log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2: Let f (z) be analytic for $|z| \le R, f(0) \ne 0$ and $|f(z)| \le M$ for $|z| \le R$. Then the number of zeros of f(z) in $|z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 2.

IV. Proofs of Theorems

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)\\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0.\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, j = 1,......,n$, we have, by using the hypothesis,

$$|F(z)| \ge |a_n||z|^{n+1} - [|a_n - a_{n-1}||z|^n + ... + |a_1 - a_0||z| + |a_0|]$$

$$= |z|^n[|a_n||z| - \{\frac{|a_n - a_{n-1}|}{|z|} + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}]$$

$$> |z|^n[|a_n||z| - \{|a_n - a_{n-1}| + ... + |a_1 - a_0| + |a_0|\}]$$

$$= |z|^n[|a_n||z| - \sum_{j=0}^{n}|a_j - a_{j-1}|]$$

$$\ge |z|^n[|a_n||z| - |a_n|]$$

$$= |z|^n|a_n|(|z| - 1)$$

$$> 0.$$

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This shows that F(z) has no zero in $|z| > 1$. Consequently, all the zeros of F(z) lie in $|z| \le 1$.
Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in $|z| \le 1$.

Again , $F(z) = a_0 + G(z)$,
where

$$G(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z$$

So that G(z) is analytic for $|z| \le R$, G(0)=0 and for $|z| \le R$, $R \ge 1$,

$$|G(z)| \le |a_n| R^{n+1} + |a_n - a_{n-1}| R^n + ... + |a_1 - a_0| R + |a_0| - |a_0|$$

$$\le |a_n| R^{n+1} + R^n [|a_n - a_{n-1}| + ... + |a_1 - a_0| + |a_0|] - |a_0|$$

$$\le |a_n| R^{n+1} + R^n |a_n| - |a_0|$$

$$= |a_n| R^n (R+1) - |a_0|$$

and for $|z| \le R$, $R \le 1$,

$$|G(z)| \le |a_n| R^{n+1} + R|a_n| - R|a_0|$$

$$= |a_n| R(R^n + 1) - |a_0|.$$

Therefore, by Schwarz Lemma, for $|z| \le R$,

$$|G(z) \le [\| a_n| R^n (R+1) - |a_0|] |z| \text{ for } R \ge 1$$

and

$$|G(z) \le [\| a_n| R(R^n + 1) - |a_0|] |z| \text{ for } R \le 1.$$

Hence, for $|z| \le R$, $R \ge 1$,

$$|F(z)| = |a_0 + G(z)|$$

$$\ge |a_0| - |G(z)|$$

$$\ge |a_0| - [|a_n| R^n (R+1) - |a_0|] |z|$$

$$> 0$$

if

$$|z| < \frac{|a_0|}{|a_n| R^n (R+1) - |a_0|}.$$

and for $|z| \le R$, $R \le 1$,

$$|F(z)| \ge |a_0| - [|a_n| R(R^n + 1) - |a_0|] |z|$$

$$> 0$$

if

$$|z| < \frac{|a_0|}{|a_n| R(R^n + 1) - |a_0|}.$$

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{|a_n| R^n (R+1) - |a_0|}$ for $R \ge 1$ and F(z) has no zero in $|z| < \frac{|a_0|}{|a_n| R(R^n + 1) - |a_0|}$ for $R \le 1$.

Consequently, all the zeros of F(z) and hence P(z) lie in $|z| \ge \frac{|a_0|}{|a_n| R^n (R+1) - |a_0|}$ for $R \ge 1$ and in $|z| \ge \frac{|a_0|}{|a_n| R(R^n + 1) - |a_0|}$ for $R \le 1$.

That completes the proof of Theorem 2.

Proof of Theorem 3. For the polynomial F(z) in the proof of Theorem 2, we have, by using the hypothesis, for $|z| \le R$, $R \ge 1$,

$$|F(z)| \le |a_n| R^{n+1} + |a_n - a_{n-1}| R^n + ... + |a_1 - a_0| R + |a_0|$$

$$\le |a_n| R^{n+1} + R^n [|a_n - a_{n-1}| + ... + |a_1 - a_0| + |a_0|]$$

$$\le |a_n| R^{n+1} + R^n |a_n|$$

$$= |a_n| R^n (R+1)$$

and for $|z| \le R$, $R \le 1$,

$$|F(z)| \le |a_n| R^{n+1} + R|a_n| + (1-R)|a_0|$$

$$= |a_n| R(R^n + 1) + (1-R)|a_0|.$$

Since F(z) is analytic for $|z| \le R$ and $F(0) = a_0 \ne 0$, it follows by Lemma 2 that the number of zeros of F(z) in $|z| \le \frac{R}{k}, k > 1$ does not exceed

$$\frac{1}{\log k} \log \frac{|a_n| R^n (R+1)}{|a_0|} \text{ for } R \ge 1$$

and

$$\frac{1}{\log k} \log \frac{|a_n| R(R^n + 1) + (1-R)|a_0|}{|a_0|} \text{ for } R \le 1.$$

Since the zeros of P(z) are also the zeros of F(z), the proof of Theorem 3 is complete.

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