Bounds for the Moduli of Zeros of Polynomials

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Abstract—In this paper we find bounds for the moduli of the zeros of a polynomial in terms of its coefficients. The results so obtained generalize many results known already in the field.

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I. INTRODUCTION

A classical result which gives a bound for the moduli of all the zeros of a polynomial in terms of its coefficients is the following known as Cauchy’s Theorem [2,4]:

Theorem A. All the zeros of the polynomial

\[ P(z) = \sum_{j=0}^{n} a_j z^j \]

of degree \( n \) lie in the circle \( |z| < 1 + M \),

where \( M = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right| \).

Another elegant classical result giving a bound for the moduli of all the zeros of a polynomial with real coefficients is the following known as the Enestrom-Kakeya Theorem [2,4]:

Theorem B. Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[ a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0. \]

Then all the zeros of \( P(z) \) lie in \( |z| \leq 1 \).

Regarding the number of zeros of the polynomial in Theorem B in a smaller disc than the closed unit disc, Q.G.Mohammad [3] proved the following result:

Theorem C. The number of zeros of the polynomial \( P(z) \) of degree \( n \) such that

\[ \sum_{j=0}^{n-1} |a_j| \leq |a_n| \]

does not exceed \( \frac{1}{\log 2} \log \frac{a_n}{a_0} \).

II. MAIN RESULTS

The aim of this paper is to weaken the hypothesis in Theorem D and prove

Theorem 1. Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients satisfying

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|, a_{-1} = 0. \]

Then all the zeros of \( P(z) \) lie in \( \frac{|a_0|}{2|a_n| - |a_0|} \leq |z| \leq 1 \).

Remark 1. If we consider the polynomial

\[ P(z) = 6z^2 + 5z + 1, \]

then Theorems B and C are not applicable. By Theorem A, all the zeros of \( P(z) \) lie in \( |z| < 1.83 \), whereas by Theorem 1, all the zeros of \( P(z) \) lie in \( |z| \leq 1 \). In fact, the zeros of \( P(z) \) are \( \frac{1}{2} \) and \( \frac{1}{3} \) whose moduli are less than 1, which is less than the Cauchy’s bound.

We prove a more general result than Theorem 1 as follows:

Theorem 2. Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients satisfying

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|, a_{-1} = 0. \]

Then all the zeros of \( P(z) \) lie in

\[ \frac{|a_0|}{a_n |R^n (R + 1) - |a_0|} \leq |z| \leq 1 \]

for \( R \geq 1 \)

and in

\[ \frac{|a_0|}{a_n |R^n + 1 - |a_0|} \leq |z| \leq 1 \]

for \( R \leq 1 \).

Remark 2. For \( R=1 \), Theorem 2 reduces to Theorem 1.

Theorem 3. Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients satisfying

\[ \sum_{j=0}^{n} |a_j| \leq |a_n| \]

then \( P(z) \) has all its zeros in \( |z| \leq 1 \).

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|, a_{-1} = 0. \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{R}{k}, k > 1, R \geq 1 \) does not exceed

\[ \frac{1}{\log k} \log \frac{|a_n| R^n (R + 1)}{|a_0|} \]

and the number of zeros of \( P(z) \) in \( |z| \leq \frac{R}{k}, k > 1, 0 < R \leq 1 \) does not exceed

\[ \frac{1}{\log k} \log \frac{|a_n| R(R^{n+1} + 1)(1 - R)|a_0|}{a_0}. \]

Taking \( R=1 \) in Theorem 3, we get the following result:

**Corollary 1.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients satisfying

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|, a_{-1} = 0. \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{k}, k > 1 \) does not exceed

\[ \frac{1}{\log k} \log \frac{2|a_n|}{|a_0|}. \]

Taking \( k=2 \) in Cor.1, we get the following result:

**Corollary 2.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients satisfying

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|, a_{-1} = 0. \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[ \frac{1}{\log 2} \log \frac{2|a_n|}{|a_0|}. \]

If also the coefficients \( a_j, j = 0, 1, \ldots, n \) are all real and positive, we get the following result from Cor.2:

**Corollary 3.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with real positive coefficients satisfying

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|, a_{-1} = 0. \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[ 1 + \log \frac{|a_n|}{|a_0|}. \]

**Remark 3.** If \( a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0 \), then

\[ \sum_{j=0}^{n} |a_j - a_{j-1}| = |a_n| = |a_0| = a_0 \]

and we get Theorem C from Cor.3.

**III. LEMMAS**

For the proof of Theorem 3, we need the following results:

**Lemma 1:** Let \( f(z) \) (not identically zero) be analytic for \( |z| \leq R, f(0) \neq 0 \) and \( f(a_k) = 0, k = 1, 2, \ldots, n \). Then

\[ \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)| = \sum_{j=1}^{n} \log \left| \frac{R}{|a_j|} \right|. \]

Lemma 1 is the famous Jensen’s Theorem (see page 208 of [1]).

**Lemma 2:** Let \( f(z) \) be analytic for \( |z| \leq R, f(0) \neq 0 \) and \( |f(z)| \leq M \) for \( |z| \leq R \). Then the number of zeros of \( f(z) \) in \( |z| \leq \frac{R}{c}, c > 1 \) does not exceed

\[ \frac{1}{\log c} \log \frac{M}{|f(0)|}. \]

Lemma 2 is a simple deduction from Lemma 2.

**IV. PROOFS OF THEOREMS**

**Proof of Theorem 2.** Consider the polynomial

\[ F(z) = (1 - z)P(z) = (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0) \]

\[ = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0. \]

For \( |z| > 1 \) so that \( \frac{1}{|z|^j} < 1, j = 1, \ldots, n \), we have, by using the hypothesis,

\[ |F(z)| \geq |a_n| |z|^{n+1} - |a_n - a_{n-1}| |z|^n + \ldots + |a_1 - a_0| |z| + |a_0| \]

\[ = |a_n|^n |z| - \frac{|a_n - a_{n-1}|}{|z|^j} + \ldots + \frac{|a_1 - a_0|}{|z|} + |a_0| \]

\[ > |a_n|^n |z| - |a_n - a_{n-1}| + \ldots + |a_1 - a_0| + |a_0| \]

\[ = |a_n|^n |z| - \sum_{j=0}^{n} |a_j - a_{j-1}| \]

\[ \geq |a_n|^n |z| - |a_n| \]

\[ = |a_n|^n |z| - 1 \]

\[ > 0. \]
This shows that F(z) has no zero in $|z| > 1$. Consequently, all the zeros of F(z) lie in $|z| \leq 1$.

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in $|z| \leq 1$.

Again, $F(z) = a_0 + G(z)$, where

$$G(z) = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \ldots + (a_1 - a_0) z$$

So that G(z) is analytic for $|z| \leq R$, $G(0)=0$ and for $|z| \leq R$, $R \geq 1$.

$$|G(z)| \leq |a_n| R^{n+1} + |a_{n-1}| R^n + \ldots + |a_1 - a_0| + |a_n| - |a_0|$$

and for $|z| \leq R$, $R \geq 1$,

$$|G(z)| \leq |a_n| R^n (R + 1) - |a_0|$$

Therefore, by Schwarz Lemma, for $|z| \leq R$,

$$|G(z)| \leq \|a_n| R^n (R + 1) - |a_0|\| |z|$$

for $R \geq 1$

$$|F(z)| = \|a_0 + G(z)\| \geq |a_0| - |G(z)| \geq |a_0| - |a_n| R^n (R + 1) - |a_0| |z|$$

if

$$|z| \leq \frac{|a_0|}{|a_n| R^n (R + 1) - |a_0|}.$$

and for $|z| \leq R$, $R \leq 1$,

$$|F(z)| \geq \|a_0 - |a_n| R^n (R + 1) - |a_0| |z|$$

if

$$|z| \leq \frac{|a_0|}{|a_n| R^n (R + 1) - |a_0|}.$$

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{|a_n| R^n (R + 1) - |a_0|}$ for $R \geq 1$ and F(z) has no zero in $|z| < \frac{|a_0|}{|a_n| R^n (R + 1) - |a_0|}$ for $R \leq 1$.

Consequently, all the zeros of F(z) and hence P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_n| R^n (R + 1) - |a_0|}$$

for $R \geq 1$ and in

$$|z| \geq \frac{|a_0|}{|a_n| R^n (R + 1) - |a_0|}$$

for $R \leq 1$.

That completes the proof of Theorem 2.

**Proof of Theorem 3.** For the polynomial F(z) in the proof of Theorem 2, we have, by using the hypothesis, for $|z| \leq R$, $R \geq 1$,

$$|F(z)| \leq |a_n| R^{n+1} + |a_{n-1}| R^{n-1} + \ldots + |a_0|$$

and for $|z| \leq R$, $R \leq 1$,

$$|F(z)| \leq |a_n| R^{n+1} + |a_{n-1}| R^{n-1} + \ldots + |a_0|$$

Since F(z) is analytic for $|z| \leq R$ and $F(0) = a_0 \neq 0$, it follows by Lemma 2 that the number of zeros of F(z) in $|z| \leq \frac{R}{k}$, $k > 1$ does not exceed

$$\frac{1}{\log k} \log \frac{|a_n| R^n (R + 1)}{|a_0|}$$

for $R \geq 1$ and

$$\frac{1}{\log k} \log \frac{|a_n| R^n (R + 1) + (1-R)|a_0|}{|a_0|}$$

for $R \leq 1$.

Since the zeros of P(z) are also the zeros of F(z), the proof of Theorem 3 is complete.

**REFERENCES**


