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Zeros of a Polynomial with Restricted Coefficients

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Abstract*— *In this paper we consider a class of polynomials whose coefficients satisfy certain conditions and locate the regions containing all their zeros.

Mathematics Subject Classification: *30C10, 30C15.*

Keywords*— *Coefficients, polynomial, zeros.

I. Introduction

The following theorem known as the Enestrom-Kakeya Theorem [8], [9] is of great importance in the theory of distribution of zeros of a polynomial:

Theorem A: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$.

Then all the zeros of P(z) lie in $|z| \le 1$.

A lot of generalizations and extensions of this result are available in the literature [1-10]. Recently Gulzar et al [6] proved the following such result:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j, \operatorname{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \tau\alpha_\lambda$

and

$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|$.

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k-1)\alpha_n}{a_n}\right| \le \frac{k\alpha_n-\tau\alpha_\lambda+(1-\tau)|\alpha_\lambda|+L+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

II. Main Results

In this paper we prove the following result:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda,\mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1,k_2 \le 1; \tau_1,\tau_2 \ge 1$,

$k_1\alpha_n \le \alpha_{n-1} \le ...... \le \tau_1\alpha_\lambda$

$k_2\beta_n \le \beta_{n-1} \le ...... \le \tau_2\beta_\mu$,

and

$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|$,

$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|$,

Then all the zeros of P(z) lie in

$$\left|z-\frac{(1-k_1)\alpha_n+i(1-k_2)\beta_n}{a_n}\right| \le \frac{1}{|a_n|}[\tau_1(\alpha_\lambda+|\alpha_\lambda|)$$
$$+\tau_2(\beta_\mu+|\beta_\mu|)-|\alpha_\lambda|-|\beta_\mu|-k_1\alpha_n-k_2\beta_n+L+M].$$

For different values of the parameters, we get many interesting results. For example**,** if we take a_j real i.e. $\beta_j=0, \forall j=0,1,.....,n; k_1=k, \tau_1=\tau$, then we get the following result from Theorem 1:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $\lambda,; 0 \le \lambda \le n-1$ and for some $k \le 1; \tau \ge 1$,

$ka_n \le a_{n-1} \le ...... \le \tau a_\lambda$

and

$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|$,

Then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{\tau(|a_\lambda|+a_\lambda)-ka_n-|a_\lambda|+L.}{|a_n|}.$$

If we take $\tau_1=\tau_2=1$ in Theorem 1, we get the following:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda,\mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1,k_2 \le 1$, ,

$k_1\alpha_n \le \alpha_{n-1} \le ...... \le \alpha_\lambda$

$k_2\beta_n \le \beta_{n-1} \le ...... \le \beta_\mu$,

and

$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|$,

$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|$,

Then all the zeros of P(z) lie in

$$\left|z-\frac{(1-k_1)\alpha_n+i(1-k_2)\beta_n}{a_n}\right| \le \frac{\alpha_\lambda+\beta_\mu)-k_1\alpha_n-k_2\beta_n+L+M.}{|a_n|}.$$

If we take $k_2=\tau_2=1$ in Theorem 1, we get the following:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$,

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$j=0,1,2,......,n$ such that for some $\lambda,\mu; 0\le\lambda\le n-1, 0\le\mu\le n-1$ and for some $k_1\le 1;\tau_1\ge 1$,

$k_1\alpha_n\le\alpha_{n-1}\le......\le\tau_1\alpha_\lambda$

$\beta_n\le\beta_{n-1}\le......\le\beta_\mu$,

and

$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|$,

$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|$,

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{\tau_1(|\alpha_\lambda|+\alpha_\lambda)+\beta_\mu-k_1\alpha_n-\beta_n-|\alpha_\lambda|+L+M.}{|a_n|}$$

If we take $k_1=k_2=\tau_1=\tau_2=1$ in Theorem 1, we get the following:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$, $\text{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda,\mu; 0\le\lambda\le n-1, 0\le\mu\le n-1$,

$\alpha_n\le\alpha_{n-1}\le......\le\alpha_\lambda$

$\beta_n\le\beta_{n-1}\le......\le\beta_\mu$,

and

$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|$,

$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|$,

Then all the zeros of P(z) lie in

$$|z|\le\frac{\alpha_\lambda+\beta_\mu-\alpha_n-\beta_n+L+M}{|a_n|}.$$

III. PROOF OF THEOREM 1

Consider the polynomial

$$\begin{aligned}
F(z)&=(1-z)P(z)\\
&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\
&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}+(a_\lambda-a_{\lambda-1})z^\lambda\\
&\quad+......+(a_1-a_0)z+a_0\\
&=-a_nz^{n+1}-(k_1-1)\alpha_nz^n+(k_1\alpha_n-\alpha_{n-1})z^n\\
&\quad+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}......+(\alpha_{\lambda+1}-\tau_1\alpha_\lambda)z^{\lambda+1}\\
&\quad+(\tau_1-1)\alpha_\lambda z^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})z^\lambda+......+(\alpha_1-\alpha_0)z\\
&\quad+\alpha_0+i\{(k_2\beta_n-\beta_{n-1})z^n-(k_2-1)\beta_nz^n+......+(\beta_{\mu+1}-\tau_2\beta_\mu)z^{\mu+1}\\
&\quad+(\tau_2-1)\beta_\mu z^{\mu+1}+(\beta_\mu-\beta_{\mu-1})z^\mu+......+(\beta_1-\beta_0)z+\beta_0\}
\end{aligned}$$

For $|z|>1$ so that $\frac{1}{|z|^j}<1,\forall j=1,2,......,n$, we have, by using the hypothesis

$$\begin{aligned}
|F(z)|&\ge|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n||z|^n\\
&\quad-[|k_1\alpha_n-\alpha_{n-1}||z|^n+|\alpha_{n-1}-\alpha_{n-2}||z|^{n-1}......+|\alpha_{\lambda+1}-\tau_1\alpha_\lambda||z|^{\lambda+1}\\
&\quad+|\tau_1-1||\alpha_\lambda||z|^{\lambda+1}+|\alpha_\lambda-\alpha_{\lambda-1}||z|^\lambda\\
&\quad+......+|\alpha_1-\alpha_0||z|+|\alpha_0|+\{|k_2\beta_n-\beta_{n-1}||z|^n\\
&\quad+|\beta_{n-1}-\beta_{n-2}||z|^{n-1}......+|\beta_{\mu+1}-\tau_2\beta_\mu||z|^{\mu+1}\\
&\quad+|\tau_2-1||\beta_\mu|^\mu+|\beta_\mu-\beta_{\mu-1}||z|^{\mu-1}+......+|\beta_1-\beta_0||z|+|\beta_0|]\\
&=|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|-\{|k\alpha_n-\alpha_{n-1}|\\
&\quad+\frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|}+......\\
&\quad+\frac{|\alpha_{\lambda+1}-\tau_1\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{(\tau_1-1)|\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{|\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}}\\
&\quad+......+\frac{|\alpha_1-\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}\\
&\quad+|k_2\beta_n-\beta_{n-1}|+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+......+\frac{|\beta_{\mu+1}-\tau_2\beta_\mu|}{|z|^{n-\mu-1}}\\
&\quad+\frac{(\tau_2-1)|\beta_\mu|}{|z|^{n-\mu}}+......+\frac{|\beta_1-\beta_0|}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\}]\\
&=|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|-\{|k\alpha_n-\alpha_{n-1}|\\
&\quad+|\alpha_{n-1}-\alpha_{n-2}|+......\\
&\quad+|\alpha_{\lambda+1}-\tau_1\alpha_\lambda|+(\tau_1-1)|\alpha_\lambda|+|\alpha_\lambda-\alpha_{\lambda-1}|\\
&\quad+......+|\alpha_1-\alpha_0|+|\alpha_0|\\
&\quad+|k_2\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+......+|\beta_{\mu+1}-\tau_2\beta_\mu|\\
&\quad+(\tau_2-1)|\beta_\mu|+|\beta_\mu-\beta_{\mu-1}|+......+|\beta_1-\beta_0|+|\beta_0|\}]\\
&=|z|^n[|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|\\
&\quad-\{\alpha_{n-1}-k_1\alpha_n+\alpha_{n-2}-\alpha_{n-1}\\
&\quad+......+\tau_1\alpha_\lambda-\alpha_{\lambda+1}+(\tau_1-1)|\alpha_\lambda|+|\alpha_\lambda-\alpha_{\lambda-1}|\\
&\quad+......+|\alpha_1-\alpha_0|+|\alpha_0|+\beta_{n-1}-k_2\beta_n+\beta_{n-2}-\beta_{n-1}\\
&\quad+......+\tau_2\beta_\mu-\beta_{\mu+1}+(\tau_2-1)|\beta_\mu|+|\beta_\mu-\beta_{\mu-1}|\\
&\quad+......+|\beta_1-\beta_0|+|\beta_0|\}]\\
&=|z|^n[|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|\\
&\quad-\{\tau_1(\alpha_\lambda+|\alpha_\lambda|)+\tau_2(\beta_\mu+|\beta_\mu|)\\
&\quad-|\alpha_\lambda|-|\beta_\mu|-k_1\alpha_n-k_2\beta_n+L+M\}]\\
&>0
\end{aligned}$$

if

$$|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|>\tau_1(\alpha_\lambda+|\alpha_\lambda|)$$
$$+\tau_2(\beta_\mu+|\beta_\mu|)-|\alpha_\lambda|-|\beta_\mu|-k_1\alpha_n-k_2\beta_n+L+M$$

i.e. if

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$$\left| z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n} \right| \le \frac{1}{|a_n|}[\tau_1(\alpha_\lambda + |\alpha_\lambda|)$$
$$+\tau_2(\beta_\mu + |\beta_\mu|) - |\alpha_\lambda| - |\beta_\mu| - k_1\alpha_n - k_2\beta_n + L + M].$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n} \right| \le \frac{1}{|a_n|}[\tau_1(\alpha_\lambda + |\alpha_\lambda|)$$
$$+\tau_2(\beta_\mu + |\beta_\mu|) - |\alpha_\lambda| - |\beta_\mu| - k_1\alpha_n - k_2\beta_n + L + M].$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left| z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n} \right| \le \frac{1}{|a_n|}[\tau_1(\alpha_\lambda + |\alpha_\lambda|) \quad ,$$
$$+\tau_2(\beta_\mu + |\beta_\mu|) - |\alpha_\lambda| - |\beta_\mu| - k_1\alpha_n - k_2\beta_n + L + M].$$

That proves Theorem 1.

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