International Research Journal of Advanced Engineering and Science
ISSN: 2455-9024

A Simple Study on the Structures of Zeroid and Ordered Semirings

N. Sulochana[1], M. Amala[2], Y. Monikarchana[3]
[1]Dept of Mathematics, K.S.R.M College of Engineering, Kadapa, Andhra Pradesh, India
[2]Dept. of Applied Mathematics, Yogi Vemana University, Kadapa, Andhra Pradesh, India
[3]Dept of Mathematics, Sree Vidyanikethan Engineering College, Tirupati, Andhra Pradesh, India

Abstract*— *In this paper we have proved that Suppose S is a semiring in which (S, +) be zeroid then a + ab + b = 2b in the following cases. (i) If (S, ·) is right singular semigroup and (ii) If (S, ·) is a band. Also we have established that If S is a totally ordered semiring with multiplicative identity 1 and (S, •) is p.t.o, then (S, +) is non-negatively ordered.

AMS Mathematics Subject Classification (2010): 20M10, 16Y60.

Keywords*— *Band, K-regular semiring, maximum, positive totally ordered.

I. INTRODUCTION

Algebraic systems are artistic with a partially or fully ordered met within several disciplines of Mathematics. In recent years interest in the study of partially ordered and fully ordered semigroups, groups, semirings, semi modules, rings and fields has been increasing enormously. The theory of semigroups had essentially two origins. One was an attempt to generalize both group theory and ring theory to the algebraic system consisting of a single associated operation which from the group theoretical point of view omits the axioms of the existence of the identities and inverses and from the ring theoretical point of view omits the additive structure of the ring. Semirings are used for modeling Economics, social network analysis, queuing theory, computation of bio-polymers, penalty theory in artificial intelligence, computation theory, modern control theory of psychological phenomenon. Semirings are used for physical theory on cognitive processes.

Recently from three decades, there is great impact of semigroup theory and semiring theory on the development of ordered semirings both in theory and applications, which are of the same kind to ordered rings and ordered semirings. In this direction the works of H. J. Weinert M. Satyanarayana, J. Hanumanthachari, K. Venuraju and H.J.Weinert, M. Satyanarayana, J. Hanumanthachari and D. Umamaheswara Reddy are worthful to mention. The zeroid of a semiring S as introduced by Bourne and Zassenhaus is $\{x \in S \mid z + x = z$ or $x + z = z$ for some $z \in R\}$. It follows that a left semi subtractive semiring with zero as its zeroid has right additive cancellation. In first section, the structure of semiring in which (S, +) is zeroid are given. Thought out this section unless otherwise mentioned S is a semiring in which (S, +) is a zeroid. The zeroid of a semiring is denoted by Z. In second section the importance was given to multiplicative identity '1' and few results on ordered semirings with multiplicative identity '1' are also studied.

II. PRELIMINARIES

Definition 2.1:
The zeroid of a semiring S is $\{x \in S \mid z + x = z$ or $x + z = z$ for some $z \in R\}$.

Definition 2.2:
In an additive idempotent Multiplicatively Subidempotent if $a + a^2 = a$.

Definition 2.3:
A semigroup (S, •) is left (right) singular if ab = a (ab = b) for all a, b in S.

Definition 2.4:
A semigroup (S, •) is a band if $a^2 = a$ for all a in S.

Definition 2.5:
A semiring S is totally ordered semiring (t.o.s.r) if there exists a partially order '≤ ' on S such that (i) (S, +) is a t. o. s. g and (ii) (S, •) is a t. o. s. g.
It is usually denoted by (S, +, •, ≤).

Definition 2.6:
An element a in a partially ordered semigroup (S, •, ≤) is non-negative
(non-positive) if $a^2 \geq a$ ($a^2 \leq a$).
A partially ordered semigroup (S, •, ≤) is non-negatively ordered (non-positively ordered) if every element of S is non-negative (non-positive).

Definition 2.7:
In a totally ordered semiring (S, +, •, ≤)
(i) (S, +, ≤) is positively totally ordered (p.t.o), if $a + x \geq a, x$ for all a, x in S.
(ii) (S, •, ≤) is positively totally ordered (p.t.o), if $ax \geq a, x$ for all a, x in S.
(iii) (S, +, ≤) is negatively totally ordered (n.t.o), if $a + x \leq a, x$ for all a, x in S.
(iv) (S, •, ≤) is positively totally ordered (n.t.o), if $ax \leq a, x$ for all a, x in S.

III. STRUCTURES OF ZEROID

Theorem 3.1: If S is a semiring in which Z is a zeroid and (S, ·) is band then S is k –regular semiring.

Proof: From the definition of zeroid we have, a + b = b ---- (I)
First let us consider aba = a (a + b) a
Since (S, ·) is a band $a^2 = a$ then above equation becomes aba = a + aba
Thus aba = a + aba
Therefore we can conclude that S is k – regular

N. Sulochana, M. Amala, and Y. Monikarchana, "A simple study on the structures of zeroid and ordered semirings," *International Research Journal of Advanced Engineering and Science*, Volume 1, Issue 4, pp. 26-28, 2016.

Proposition 3.2: Let S is a semiring. If $a \in Z$ then $a^n \in Z$
Proof: Suppose $a \in Z$. Then a + b = b or b + a = b -----(I)
Let us consider a + b = b
This implies $a^2 + ab = ab$
$\Rightarrow a^2 + ab + b^2 = ab + b^2$
$\Rightarrow a^2 + (a + b) b = (a + b) b$
$\Rightarrow a^2 + b^2 = b^2$
$\Rightarrow a^3 + ab^2 = ab^2$
$\Rightarrow a^3 + ab^2 + b^3 = ab^2 + b^3$
$\Rightarrow a^3 + (a + b) b^2 = (a + b) b^2$
$\Rightarrow a^3 + b^3 = b^3$
Proceeding in the similar way it results in $a^n + b^n = b^n$
Therefore $a^n \in Z$

Theorem 3.3: Let S is a semiring. If $a \in Z$ and (S, +) is right cancellative, then 'a' is multiplicatively subidempotent element.
Proof: Let us take $a \in Z$, a + b = b or b + a = b -------- (I)
Suppose a + b = b
$\Rightarrow a^2 + ab = ab$
$\Rightarrow a + a^2 + ab = a + ab$
By using right cancellation it becomes $a + a^2 = a$
Thus S is multiplicatively subidempotent

Proposition 3.4: Suppose S is a semiring in which (S, +) be zeroid then a + ab + b = 2b in the following cases
(i) If (S, ·) is right singular semigroup.
(ii) If (S, ·) is a band.
Proof: (i) Let $a \in Z$ where Z is a zeroid of S. then there exists $b \in S$ such that a + b = b or
b + a = b
Suppose (S, ·) is right singular we have ab = b for all a, b in S
Let us consider ab = b
By adding 'b' on both sides we get a + ab = a + b
$\Rightarrow$ a + ab = b
Again on addition of 'b' on both sides we obtain a + ab + b = b + b
Thus a + ab + b = 2b for all a, b in S
(ii) Suppose (S, ·) is a band then $a^2 = a$ for all a in S
Again let us take a + b = b
$\Rightarrow a + b^2 = b$
$\Rightarrow$ a + (a + b) b + b = b
$\Rightarrow a + ab + b^2 = b$
$\Rightarrow$ a + ab + b = 2b
Hence a + ab + b = 2b for all a, b in S

Theorem 3.5: Let S be a semiring satisfying the identity a + b + ab = b. If (S, +) is zeroid and left cancellative, then (S, ·) is a band.
Proof: First suppose that S satisfying the identity a + b + ab = b -------- (1)
Let $a \in Z$ where Z is a zeroid of S then there exists b in S such that a + b = b or b + a = b
From equation (1) we obtain a + b + a (b + a) = b
$\Rightarrow a + ab + b + a^2 = b$
$\Rightarrow b + a^2 = b + a$
By applying left cancellation law we get $a^2 = a$ -------- (2)

IV. STRUCTURES OF ORDERED SEMIRING

Theorem 4.1: Let S be a totally ordered semiring with multiplicative identity 1. If S satisfies a + x + ax = ax for all a, x in S, then $x \geq 1 + x$ for all x in S.
Proof: By hypothesis S is totally ordered semiring with multiplicative identity 1
then $1 + 1 \geq 1$ or $1 + 1 \leq 1$
$\Rightarrow x (1 + 1) \geq x.1$ or $x (1 + 1) \leq x.1$
$\Rightarrow x + x \geq x$ or $x + x \leq x$
Suppose a = 1 in a + x + ax = ax then $1 + x + 1.x = 1.x \geq 1 + x$
$\Rightarrow x \geq 1 + x$
Thus $x \geq 1 + x$ for all x in S

Proposition 4.2: If S is a totally ordered semiring with multiplicative identity 1 and (S, •) is p.t.o, then (S, +) is non-negatively ordered.
Proof: By hypothesis (S, •) is positively totally ordered then 1 is the minimum element
So $1 + 1 \geq 1$
$\Rightarrow x + x \geq x$ for all x in S
Hence (S, +) is non-negatively ordered

Theorem 4.3: If S is a totally ordered semiring with multiplicative identity '1' and if there exists two elements a and x in S satisfying ax (a + x + ax) = ax and (S, •) is p.t.o, then (S, +) is non-positively ordered.
Proof: By hypothesis ax = ax (a + x + ax)
$ax = axa + ax^2 + ax. ax$
Since (S, •) is positively totally ordered implies $ax \geq ax + ax + ax$
$\Rightarrow ax + ax \geq ax + ax + ax + ax$
$\Rightarrow 2ax \geq 4ax$
$\Rightarrow$ 2ax is non-positively ordered
Therefore (S, +) is non-positively ordered

Proposition 4.4: Suppose S is a totally ordered semiring satisfying a + x + ax = ax for all
a, x in S and (S, +) is non-negatively ordered. Then (S, +) is a band.
Proof: Suppose a = x = 1 in a + x + ax = ax and using (S, +) non-negatively ordered we get
$1.1 = 1 + 1 + 1.1 \geq 1 + 1$
$\Rightarrow 1 \geq 1 + 1 \rightarrow$ **(1)**
Also (S, +) is non-negatively ordered then $1 + 1 \geq 1 \rightarrow$ **(2)**
From equations (1) and (2) we obtain 1 + 1 = 1
$\Rightarrow$ b + b = b for all b in S
Hence (S, +) is a band

REFERENCES

[1] S. Bourne and H. Zassenhaus, "On the semi-radical of a semiring," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 44, issue 9, pp. 907-914, 1958.
[2] S. Ghosh, "Another note on the least lattice congruence on semirings" *Soochow Journal of Mathematics*," vol. 22, no. 3, pp. 357-362, 1996.

[3] J. S. Golan, *The Theory of Semirings with Applications in Mathematics and Computer Science*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 54, Zongamann House, Burnt Mill, Harlow, Essex CM 20 JE. England: Longman Scientific and Technical, 1992.
[4] J. S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, 1999.
[5] J. S. Golan, *Semirings and Affine Equations over Them: Theory and Applications*, Kluwer Academic Publishers (1999).
[6] J. Hanumanthachari, K. Venuraju, and H. J. Weinert, "Some results on partially ordered semirings and semigroups," Algebra order, Proc. of the International Symposium on ordered Algebraic Structures, Luminy – Marseilles, 1984.
[7] M. Satyanarayana, "Positively ordered semigroups," Lecture notes in Pure and Applied Mathematics, Marcel Dekker, Inc., vol. 42, 1979.
[8] M. Satyanarayana, "Totally ordered archimedean semigroups," *Semigroup Forum*, vol. 18, pp. 265-270, 1979.
[9] M. Satyanarayana, "On the additive semigroup structure of semirings," *Semigroup Forum*, vol. 23, pp. 7-14, 1981.
[10] M. Satyanarayana. "On the additive semigroup of ordered semirings," *Semigroup Forum*, vol. 31, vol. 193-199, 1985.
[11] M. Satyanarayana, J. Hanumanthachari, and D. Umamaheswara Reddy, "On the additive structure of a class of ordered semirings," *Semigroup Forum*, vol. 33, pp. 251-255, 1986.
[12] M. Satyanarayana, J. Hanumanthachari, and D. Umamaheswara Reddy, "On the multiplicative structure of a class of ordered semirings," *Semigroup Forum*, vol. 35, pp. 175-180, 1987.
[13] H. J. Weinert, "Related representation theorems for rings, semirings, near rings and semi–near rings by partial transformations and partial endomorphisms," *Proceedings of the Edinburgh Mathematical Society*, vol. 20, pp. 307-315, 1976-77.
[14] H. J. Weinert, "Multiplicative cancellativity of Semirings and Semigroups," *Acta Mathematica Academicae Scientiarum Hungaricae Tomus*, vol. 35, issue 3-4, pp. 335-338, 1980.
[15] H. J. Weinert, "Partially ordered semirings and semigroups," Algebra and order, *Proc. of the first International Symposium on ordered Algebraic structures*, Luminy–Marseilles, 1984.